Implementation of Principles of Pure Shape Dynamics in Quantum Field Theory

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Summary

Relational physics offers a parallel framework to conventional physics by eliminating all references to the background structures inherent in the Newtonian paradigm. Contemporary relational physics was originally developed in the dynamical setting. Pure Shape Dynamics (PSD) is the mathematical formulation that allows for the construction of various dynamical systems that embody those principles. It has so far been applied to Newtonian N-body mechanics, non-relativistic de Broglie–Bohm quantum mechanics, and dynamical geometry, in particular, Einstein's General Relativity. This work explores the extension of this framework to Quantum Field Theory (QFT). The principles underlying a PSD formulation of QFT are discussed, and two concrete models, using fermionic and bosonic fields respectively, are presented. Remarkably, it is shown that PSD can address the conceptual deficiencies of standard QFT, most notably the problem of UV divergences. The PSD conceptualization of time leads to a naturally regularized and well-defined Quantum Field Theory. These results are expected to pave the way for a more general Pure Shape Dynamics approach to Quantum Gravity.

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Notation

$a, b, c, \cdots$	Spatial/General indices
$A, B, C, \cdots$	Groups of double spatial indices, i.e., $A = (ab), \cdots$
$i, j, k, \cdots$	Indices enumerating objects from 1 to 3
$\alpha, \beta, \gamma, \cdots$	Spinor indices
$\mathcal{Q}$	Configuration space in the Newtonian framework
$\mathcal{P}$	Phase space, defined as cotangent space T^*Q
$\mathcal{S}$	Shape space, i.e., $\mathcal{Q}$ quotiented by symmetries and scale transformations
$\hat{O}$	Operator defined on a Hilbert space
G	Lie group
$\mathfrak{g}$	Lie algebra
$\mathrm{E}(N)$	Euclidean group in N dimensions
$\mathbb{R}^+$	Group of scale transformations
$\mathrm{Sim}(N)$	Similarity group, i.e., $\mathrm{Sim}(N) = \mathbb{R}^+ \ltimes \mathrm{E}(N)$
$\{\cdot,\cdot\}$	Poisson bracket
$\{\cdot,\cdot\}_{\mathrm{DB}}$	Dirac bracket
$[\cdot,\cdot]$	Commutator in an algebra
$[\cdot,\cdot]_+$	Anti-commutator in an algebra
$f_{x,y,\ldots}$	Real-valued function defined on a manifold
$F[f]$	Functional defined on the space of functions on a manifold
$f \cdot g$	Smearing, i.e., $f \cdot g = \int d^d x \sqrt{g}\, f(x)\, g(x)$
$\int_x$	Integration over a manifold equipped with a Riemannian metric g, i.e., $\int_x \equiv \int d^d x \sqrt{g}$
$\int_F$	Integration over fermionic degrees of freedom, i.e., $\int_F \equiv \int D[\psi^\dagger] D[\psi]$

Chapter 1

Introduction

Despite the stellar empirical success of modern physics, the foundations of physics have barely progressed. Since the 1970s and completion of the Standard Model of particle physics, we have not been able to address some of our deepest conceptual questions. Arguably, at the bottom of this abyss lie the notorious conceptual issues plaguing Quantum Mechanics (QM), possibly among other problems.

I argue that without progress at a deep conceptual level, there is no progress at all, not even empirically if we limit scientific enterprise merely to building models to capture certain phenomena. Other than historical testimony, I would like to argue more generally from human conceptualization. What follows will then serve as a *methodological* argument for significance of foundational work.

Human thinking is hierarchical. Each level within this hierarchy determines the prioritization and meaning of perceptions at the next level. If on one end, we simply place perception of factual states of affairs, then there has to be a higher level. The next level has to determine how to prioritize our focus, group and structure our factual perceptions, and place them in a meaningful narrative, and finally, *understand* them. By the same token, we can go further and further up this ladder and build up the hierarchy according to which human faculty functions.

At some point along this ladder, I propose to place scientific models which we create to capture certain phenomena once they are identified according to some higher-level priority. However, in order to even create models, human faculty needs more basic theoretical ingredients and toolkit for conceptualization to have proper conception of elements that build up a model and their relations. Without this upper supervision, even when it appears to operate implicitly, one cannot erect a model on the same level.

To make this analysis concrete, consider how various classical models are situated

with relation to one another, and together with classical mechanics above. For instance, the N-body system, which is applicable to celestial mechanics, the continuum models, applicable to hydrodynamics and aerodynamics, and the rigid-body mechanics, applicable to statics and mechanical engineering, were all developed within Newtonian framework. These models belong to distinct areas, and capture utterly distinct phenomena. However, the very basic theoretical elements and principles employed in conceptualizing them are under the guidance of Newtonian framework. One wonders how the plethora of all of these specific models could have been built in absence of this higher level guidance.

If, up to this point, we accept this hierarchical structure from facts to general theoretical frameworks, I argue that the next level is when we delve deeper into the conceptual foundation of our theories. Accordingly, the questions that arise are about how we essentially see the general aspects of the universe. How do we understand the spatial relations and the temporal structure(s)? And, what is the universe composed of as we see it? These questions are usually left behind by physicists and commonly labeled as "philosophical". However, I propose, similarly to many physicists in history, that these fundamental questions must be engaged in by the physics community. Furthermore, I argue that they are an extension of the same theorization which one comes across inevitably along the pyramid described above. As Poincaré said in *Science and Hypothesis* [1],

> *"...Only these are unconscious preconceived ideas, a thousand times more dangerous than the others."*

One ought to be aware of one's principles, an no one is without one.

In light of this proposal, it is reasonable to assume that our lack of clear understanding of foundations of modern physics has been a major obstruction in making the progress we hoped for. One such dark corner in modern physics is QM, and especially its field version that sets the underpinning of Standard Model. In this work, we shall aim to shed some light on some of obscurities of Quantum Field Theory (QFT) framework in light of a recent program, called "Pure Shape Dynamics" (PSD) [2]. PSD is essentially an adaptable formulation of Shape Dynamics (SD), which is a contemporary program for implementing relational physics.

Relationalism, as a general conceptual perspective, is almost as old as modern science itself. In parallel with Newton's completion of the first theoretical physics framework, Leibniz sparked debates about the very conception of space and time within the Newtonian framework. Newton envisaged space and time as absolute, independently existing entities, and separate from material structures, against which

one could define absolute geometrical properties. Since at least the late 19th century, however, the notion of absolutes has been abandoned. Instead, we have adopted the construction of reference frames to define spatio-temporal structures.[1] For instance, when we state Newton's second law,

$$\frac{d^2}{dt^2}\vec{r} = \vec{F}, \tag{1.1}$$

the background spatio-temporal structures are implicitly invoked in $\vec{r}$ and t. The metaphysical baggage has been left behind, but in essence, the same problem pinpointed by Leibniz persists: all knowledge of the universe ultimately boils down to relationships within the observable material universe. It is within these interrelations that one ought to meaningfully conceive of space and time. "Space is the order of co-existence, and time is the order of succession," said Leibniz in his well-known correspondence with Clarke [4].

The problem of Newton's absolutism remained unresolved as Newtonian mechanics continued to develop, and Leibniz's critique was largely ignored, owing to its conceptual nature and the absence of a mathematically constructed alternative. Several figures in the history of physics revisited the problem from new perspectives. Most notably, Mach, Poincaré, and Einstein deserve mention. For instance, Mach's principle played an important role in shaping Einstein's approach to formulating General Relativity. Yet, disappointingly, none of these efforts took root as viable alternatives to the standard framework inherited from the Newtonian tradition. Within theoretical physics, these issues remained largely unattended.

The Shape Dynamics program represents a modern relational framework. It originated in 1982 with Barbour and Bertotti's proposal for constructing relational dynamics [5]. Drawing on the works of Leibniz and Poincaré, its fundamental goal is to admit only relational properties into physical models, and properties definable in terms of the interrelations of objects that are necessary and sufficient for explaining phenomena. A crucial step in this program is the requirement that the physics of the universe be formulated in a scale-independent manner, since the overall scale is extrinsic to a closed system of interrelations.

Pure Shape Dynamics, the formulation that serves as the bedrock of this work, is a more specific implementation of Shape Dynamics ideas. It was developed in [2] for N-body systems, and the same strategy has more recently been extended to dynamical geometry and matter fields [6]. The objective is to identify a particular representation of shape space as the kinematic arena and to restrict dynamics to unfold within it

[1] See Tait's paper [3] for instance.

autonomously. This approach has the virtue of stripping away both conceptually and empirically redundant structures, thereby imposing significant constraints on physical models. In doing so, it renders the framework physically constructive.

The structure of this work is as follows. In Chapter 2, we lay out the foundational principles of the work. We will begin by summarizing the four main conceptual issues that QFT framework suffers from. We then present the principles of PSD. In Chapter 3, a purely fermionic model is constructed. As conceptually meaningful as it will be shown to be, we have empirical evidence of fermionic fields coupled to mediating bosonic fields. We show in Chapter 4 how a bosonic model can be constructed while retaining the principles applied to the fermionic models. To provide a physically relevant model, electromagnetism is studied.

Chapter 2

Conceptual Foundations

> *"If I were asked what was Christopher Columbus' greatest achievement in discovering America, my answer would not be that he took advantage of the spherical shape of the earth to get to India by the western route—this idea had occurred to others before him—or that he prepared his expedition meticulously and rigged his ships most expertly—that, too, others could have done equally well. His most remarkable feat was the decision to leave the known regions of the world and to sail westward, far beyond the point from which his provisions could have got him back home again. In science, too, it is impossible to open up new territory unless one is prepared to leave the safe anchorage of established doctrine and run the risk of a hazardous leap forward."*
>
> Werner Heisenberg [7]

2.1 Three Quantum Field Theories

Quantum Field Theory has turned into an umbrella term encompassing multiple frameworks, each constructed from different sets of principles, employing different mathematical structures, and explaining different phenomena. There are three well-established QFTs, which we will briefly describe and evaluate in terms of their relevance to our approach. It should be noted that the following is not intended as a rigorous

axiomatic presentation.

2.1.1 QFT from interacting particle physics

This approach leads to the framework that encompasses the Standard Model and describes interacting theories [8]. The fundamental ingredients and principles are as follows:

1. The configuration space of fields is defined such that the ϕ_A's satisfy

$$\phi_A : \mathbb{R}^3 \to \mathcal{T}. \tag{2.1}$$

   Here, $\mathcal{T}$ denotes the target space, which is typically taken to be $\mathbb{R}$, or more generally, a manifold such as a Lie group in the case of non-linear sigma models.

2. A self-adjoint Hamiltonian $\hat{H}$ on the Hilbert space $\mathcal{H}$ is assumed as a function of the fields ϕ_A and their conjugate momenta.

3. The fields are expanded in terms of the eigenmodes of the free Hamiltonian, a_p^A and a_p^{A*}, which are then promoted to elements of the Weyl/Clifford algebra and interpreted as annihilation and creation operators, $\hat{a}_p^A$ and $\hat{a}_p^{A\dagger}$. The vacuum state is defined by

$$\forall p \quad \hat{a}_p^A \, |0\rangle = 0. \tag{2.2}$$

4. The unitary time evolution is given by $\hat{U} = e^{-it\hat{H}}$. In practice, calculations are performed in the interaction picture with respect to the free Hamiltonian (the quadratic part) $\hat{H}_0$, where the evolution equation takes the form

$$\dot{\hat{A}}_I = -i[\hat{A}_I, \hat{H}_I - \hat{H}_0], \tag{2.3}$$

   with $\hat{A}_I = e^{-it\hat{H}_0}\hat{A}e^{it\hat{H}_0}$. Physical quantities can then be computed perturbatively by expanding the interacting part $H - H_0$ of the Hamiltonian.

The particle-physics approach to QFT has the advantage of providing an evolving, interacting theory. This allows for the modeling of various time-dependent physical phenomena, such as particle interactions and scattering processes. In particular, this framework allows for the computation of S-matrices. However, in relation to our approach, the conceptual limitation lies in its emphasis on isolating a subsystem under study from stable reference systems in the background, which define all the relevant

physical properties including energy and momentum of particles involved in scattering. From a relational standpoint, our interest lies in a QFT formulation that describes the universe as a whole. Another limitation is the reliance on perturbative methods, which technically restricts the generality of the formulation.

2.1.2 (Statistical) QFT from Condensed Matter Physics

This approach applies statistical methods to field models [9]. Systems are considered within a thermodynamic setting, where macroscopic quantities such as temperature T are externally fixed. This approach follows the principles:

1. The configuration space of fields ϕ_A is defined as

$$\phi_A : \mathbb{R}^3 \to \mathcal{T}, \tag{2.4}$$

   where $\mathcal{T}$ denotes the target space.

2. A self-adjoint Hamiltonian $\hat{H}$ is assumed.

3. The partition function is defined by

$$Z[J^A] := \operatorname{Tr} e^{-\frac{1}{T}(\hat{H}+J^A\cdot\hat{\phi}_A)}, \tag{2.5}$$

   which serves as the moment-generating functional. The thermodynamic properties as statistical functionals of the fields can then be computed using $Z[J^A]$.

Unlike the particle-physics approach, statistical problems in QFT are usually studied non-perturbatively, which allows one to establish results within this framework more generally. We aim to incorporate such techniques into our own approach. From a conceptual standpoint, however, the statistical approach is inherently tied to a background environment, as the thermodynamic assumptions require such a setting to be meaningful. This dependence is reflected in the formalism through macroscopic quantities, such as the temperature T.

2.1.3 Algebraic QFT

As part of the efforts to rigorously axiomatize QFT, a more general framework known as Algebraic Quantum Field Theory (AQFT) was developed, based purely on the algebraic structure of the theory [10, 11]. In this approach, no Hilbert space, states, or operators are postulated; instead, results are derived purely algebraically. We now present the minimal set of postulates originally formulated by Haag and Kastler [10].

1. There exists a map from open sets with compact closure, $\mathcal{B}$ (regions) in Minkowski space, to C^*-algebras[1] as

$$\mathcal{B} \to \mathcal{A}[\mathcal{B}]. \tag{2.6}$$

   These algebras constitute quasi-local "observables" within patches of spacetime.

2. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are spacelike separated (causally disjoint), we postulate that $\mathcal{A}[\mathcal{B}_1]$ and $\mathcal{A}[\mathcal{B}_2]$ commute. This is the principle of local causality.

3. "Physical states" are represented by linear forms φ over the algebras such that

$$\varphi[A^*A] \geq 0, \tag{2.7}$$

   for all A in the algebra, where A^* denotes the conjugate. This rule allows expectation values to be computed as an algebraic alternative to $\operatorname{Tr} \hat{O}\rho$ in the standard density matrix formalism, without any need to define the algebra over a vector space.

Although AQFT provides an extremely rigorous and well-defined framework for approaching QFT, it further obscures the already vague physics of the theory (see the next section). First, AQFT is formulated on a fixed background spacetime, which is incompatible with the relational approach. As will be discussed in Section 2.3, the very origin of spatio-temporal structures is the central question that PSD aims to address. Second, while the independence of AQFT from any Hilbert space representation is technically significant, it goes beyond unitary equivalence and treats representations that are unitarily inequivalent on an equal footing. This exacerbates the representation problem of QM (see Section 2.2.2). Finally, the algebraic structures posited by AQFT lack a clear physical interpretation. As Haag and Kastler noted in their paper, states and operations are defined "in terms of laboratory procedures" [10], an operationalist stance that is in stark tension with our approach, which seeks a universal, objective physical theory of interrelations within the universe (shapes).

2.2 Problems of Quantum Field Theory

QFT was a major step forward from QM: The technical problems with relativistic QM solved, QED explained, novel physical predictions such as Lamb shift verified, weak

[1] We are glossing over some of the subtleties, e.g., isotony of the map, primitiveness of the algebraic space, etc., and refer the reader to the cited paper for details.

and strong forces modeled, and the Standard Model of physics accommodated which works to this day. However, from the often-marginalized viewpoint of conceptual skepticism toward QM, it was hardly an improvement. Not only did QFT inherited the conceptual obscurities of QM, but it also aggravated the situation by creating problems of its own [12].

Depending on the level of analysis and angle, one can distill and categorize the uneasiness that QFT creates of the intellect created by QFT differently. Here, we speak of four problems, two of which are the direct result of implementation of QM itself, which carry over to QFT. They are the measurement/reality problem, the representation problem[2], the (field) interpretation problem, and the ultraviolet divergence problem. We define and explain all of them below.

2.2.1 The measurement/reality problem

The "measurement problem" has become a relatively conservative yet popular umbrella term used to encapsulate the general uneasiness one experiences upon first encountering Quantum Mechanics and attempting to make sense of its description of physical reality. Roughly speaking (a more precise definition will follow), the measurement problem refers to the inevitable dependence of the quantum-mechanical formulation on a framework consisting of a system and a "measuring" apparatus. The problem lies in the apparent inability to formulate Quantum Mechanics purely in terms of objective physical processes, independent of any act of measurement. Intuitively, one would expect that measurement should merely be a specific physical process governed by an otherwise general theoretical description.

To make this more explicit, let us recall the fundamental postulate of Quantum Mechanics, which consists of two essential rules required for making testable predictions. Using density matrices to encode the statistical information about a system, we have:

1. Rule 1: The unitary evolution $\rho \xrightarrow{U} \hat{U}\rho\hat{U}^\dagger$ governed by Schrödinger equation: $i\frac{d}{dt}\hat{U} = \hat{H}\hat{U}$.

2. Rule 2: Transition under observation, $\rho \longrightarrow \frac{M_i \rho M_i^\dagger}{\mathrm{Tr}\, M_i \rho M_i^\dagger}$ with probabilities $p_i = \mathrm{Tr}\, M_i \rho M_i^\dagger$ (Born rule). The outcomes are modeled by POVM's $M_i^\dagger M_i$'s. This is sometimes referred to as the "collaps" postulate.

The difficulty is that only Rule 1 describes an objective physical process, independent of any act of observation. However, with Rule 1 and unitary evolution alone, one

[2]Also known as the "preferred-basis problem".

cannot obtain the statistical predictions given by Rule 2, which have been empirically confirmed. Yet the inclusion of Rule 2 is precisely the source of the problem, as it introduces a dynamical transition triggered by an act of "observation". This raises the question of what precisely qualifies as an observation, and at what point it occurs for Rule 2 to take effect. Ordinarily, as in all classical theories, we would expect a physical description of observation itself, rather than placing it as a foundational element within the postulates of the theory.

To place this issue in historical context, it should be noted that the essence of the measurement problem was already present in the 1920s. Quantum "jumps" can be traced back to Bohr's transitions between quantized bound states in the "old" quantum theory. More explicitly, in his 1930 book *The Principles of Quantum Mechanics* [13], Paul Dirac discussed the essential dependence of the quantum-mechanical formulation on discontinuous changes during a measurement. As noted above, this is where the problem originates from: The failure to characterize these jumps as objective, observation-independent physical processes.

A more rigorous attempt to formalize the measurement postulate was made by John von Neumann in 1932, in his book Mathematical *Foundations of Quantum Mechanics*:

> *"We therefore have two fundamentally different types of interventions which can occur in a system S or in an ensemble. We allude here first to the abrupt changes...that are brought about by measurement, and second to the temporally graded dynamical transformations...that are generated by the energy operator (Hamiltonian) $\hat{H}$, here assumed to be time-independent."* ([14, p. 230])

This attempt, though commendable and productive in developing the formulation rigorously, led to the infamous collapse postulate. This postulate describes a dynamical process occurring during measurement, but it remains essentially distinct from the unitary dynamics that govern the system.

To state the measurement problem more precisely, we find Maudlin's classification of three measurement problems particularly instructive [15]. The classification concern how separate aspects of quantum-mechanical results are to be obtained from a linear quantum model. This is because, in principle, such a model describes superpositions of different outcomes within a single evolving wave. However, a measurement produces a single, definite outcome.

The aspects in question are, respectively: the outcome states with definite physical values; the statistics according to which these states are realized; and finally, the subsequent state of the system following a complete measurement. All three are necessarily tied to the predictions of Quantum Mechanics and fall under Rule 2. Maudlin labels the

difficulties associated with each aspect as the problem of the "outcome", the problem of "statistics", and the problem of "effect", respectively.

For reasons that will be clarified below, we deem it appropriate to group the first two problems together and shift the focus away from the problem of effect. Accordingly, we formulate the measurement problem *á la* [15]:

The Measurement Problem: The following four propositions are mutually inconsistent.

A) The wave function is *complete*, meaning that it encodes all physically knowable properties of the system.

B) The wave function evolves solely according to Rule 1; that is, via a linear evolution.

C) Measurements of the system yield unique, determinate outcomes with definite values for a set of physical properties.

D) Outcomes register with probabilities according to the Born rule.

Standard Copenhagen QM[3] resolves this contradiction by positing Rule 2, namely the measurement postulate, which supplements the linear evolution. At the cost of undermining an objective (i.e., observation-independent) physical description, the standard model sets aside Proposition B, thereby allowing for the determination of statistical outcomes while maintaining completeness of ψ. For the sake of completeness and to remain open to alternatives, it is valuable to briefly mention some of the other well-known interpretations and models of QM and how they circumvent the measurement problem as defined above.

One possibility is that Proposition A is false, implying that QM is incomplete. This view has been advocated since the inception of the theory by early critics such as Einstein, Schrödinger, and de Broglie. The so-called "hidden-variable" theories fall into this category, although some go further and challenge other propositions as well. A notable example, which will be discussed and employed in this work in Chapter 4, is the de Broglie–Bohm model [17].

Another possibility, suggested by Everett's relative-state formulation [18], now known as the "many-worlds" interpretation, opposes Proposition C. In this approach, all linear terms within the wave function are treated as equally real, corresponding to different worlds in which different outcomes are realized. Consequently, there is no unique outcome, and therefore, no problem of outcome.

[3] See [16] for a discussion of what the Copenhagen interpretation is and how it developed historically, and distinctly from Niels Bohr's original views.

Finally, we mention the GRW collapse theory [19] as an approach in which both the uniqueness of outcomes and the completeness of the wave function are retained, while the Born rule is also reproduced. In this model, the linear evolution is modified by stochastic, nonlinear jumps that generate outcomes according to Born probabilities.

At this point, we would like to highlight a broader aspect of the measurement problem. The argument is that this problem stems from a deeper issue in our conceptual understanding of Quantum Mechanics: The *reality* of the physical world is not fully acknowledged. Recognizing this fundamental problem provides a clearer context for understanding the measurement problem and clarifies what must be addressed in any attempt to resolve it.

The difficulty lies in the observation-dependent nature of the standard formulation. For a physical description to be genuinely realist, elements of the theory must correspond to presumed elements of reality, conceptualized independently of any observation. A physical description of all natural phenomena within the scope of the theory must be expressed in terms of these elements. However, the inclusion of statements about observation and measurement in the basic formulation undermines this mapping, which, by definition, must be observation-independent. Rather, it should be the role of the theory to explain what observables are and how they are to be measured. This is the point repeatedly emphasized by Einstein [20].

Therefore, we may formulate the reality problem in the manner which we believe properly construes the measurement problem:

The Measurement/Reality Problem: The following three propositions are mutually inconsistent.

A) **Completeness of the wave function:** In principle, the wave function describes all physical properties that can be determined.

B) **Physical reality:** There exist elements of physical reality onto which the wave function is mapped. These elements are presumed to be universal and independent of observations and/or the subjective knowledge of physical systems.

C) **Measurement postulate (Rule 2):** Under observation, and separately from the linear evolution, the wave function yields distinct outcomes according to Born statistics.

Let us briefly evaluate different quantum models and interpretations along these lines. Standard Copenhagen QM denies Proposition B. Furthermore, we note that the de Broglie–Bohm model violates the completeness of the wave function as well as

Proposition C, since there is no collapse postulate and the outcomes of measurement are explained differently.

We emphasize that physical reality, as understood above, and not in a metaphysical sense, serves as a guiding principle for theoreticians in constructing physical descriptions (see the brief discussion in Chapter 1). Accordingly, it should not be abandoned. Progress is possible only with the aspiration of developing a physics grounded in reality; without this guiding principle, theoretical development would be stalled.

2.2.2 The representation problem

This problem specifically arose in context of relative-state interpretation of QM [18], also known as many-worlds interpretation. The problem pertains to selection of the basis for application of Born rule. As explained in Sec. 2.2.1, QM is essentially equipped with two rules. The dynamical evolution is unitary, but the rule for statistics breaks the unitarity. Assuming the wave function $|\psi\rangle$, and the eigenvectors of some observable as $|i\rangle$'s are given, the Born probabilities are calculated as $|\langle i|\psi\rangle|^2$. This is not invariant under unitary transformation of the basis to another equivalent set $|i'\rangle = U|i\rangle$: Generally, $|\langle i'|\psi\rangle|^2 \neq |\langle i|\psi\rangle|^2$. This means in order to use standard formalism and make predictions, one has to select a *basis*. This raises the question of what basis constitutes the "right" one.

As a concrete example, note that in most QM problems one is interested in the probability distribution in configuration space. This is because the measuring apparatus that registers the results of an experiment is localized in space, and the "pointer states" (see the discussion below) correspond to eigenvectors of the position operator. However, an arbitrary linear combination of them under a unitary transformation is equally legitimate from the standpoint of QM kinematics. For instance, why do we not use bases such as $\frac{1}{\sqrt{2}}(|x\rangle + |x+\alpha\rangle)$, for arbitrary α? Yet no probability distribution of this sort has ever been empirically verified, since no spatial distribution in a superposition of states with mutually exclusive support is observable.

The fact that we need external input into the mathematical formulation of QM to select a particular basis for determining statistics is not in dispute. What is problematic is the underlying explanation for this necessity and the lack of a sufficient structure to justify the selection. The argument here is that QM, *by itself*, does not possess such a structure, and on objective grounds, it remains unclear how to make this determination, apart from relying on preference, intuition, or ad hoc prescriptions in each case.

As construed from this perspective, one might see the representation problem as part of the measurement problem. This is because as one questions what happens

during measurement, which the standard theory formally characterizes as rule 2, the states to which the system resolves is also part of the problem. However, we are essentially separating the obscurity of the measurement itself from the aspect regarding basis selection. The justification for this distinction will be explained below. We tackle these two issues separately.

It is important to mention the decoherence program here [21, 22]. It is widely believed that decoherence resolves the representation problem by ein-selecting the pointer states through the coupling of quantum systems to a macroscopic measuring apparatus regarded as the environment. The general form of the argument typically proceeds as follows.

Assume that an initial wave function $|\psi\rangle$ is assigned to a quantum system interacting with a measuring apparatus. Under time evolution, the total state of the system and environment can generally be written as

$$|\psi\rangle \rightarrow \sum_i e^{-i\epsilon_i t} |\psi_i\rangle |E_i\rangle , \tag{2.8}$$

where $|E_i\rangle$'s are orthonormal set of basis that are stable under the environment Hamiltonian. If one computes the reduced density matrix from Eq. (2.8) under the assumption $\langle E_i|E_j\rangle = \delta_{ij}$[4], it follows that

$$\rho := \mathrm{Tr}_{\mathrm{env.}} |\psi\rangle \langle\psi| = \sum_i |\psi_i\rangle \langle\psi_i| , \tag{2.9}$$

which is diagonal in $|\psi_i\rangle$-basis.This means that, in this particular basis, interference has vanished and the density matrix has reduced to a form representing "classical" statistics. This provides the justification for using these preferred bases to compute the statistics and thereby resolve the problem. Typically, for a localized apparatus with a large number of degrees of freedom and local interactions, the states $|E_i\rangle$'s describe disjoint spatial wave packets, also referred to as pointer states. These correspond to the macroscopically observed states of the measuring apparatus as it registers the outcome of a quantum measurement.

This is doubtful as a general solution, for in this argument, several presumptions are implicitly made. At least, a separation between system and environment is already assumed [23], which weakens the argument's value in the general case. It remains open

[4]A more practical formulation of decoherence program assumes disjoint supports, as a stronger condition, rather than orthogonality of the basis. That is, a representation with basis $|a\rangle$ is assumed, usually position basis, such that $\langle a|E_i\rangle\langle a|E_j\rangle \approx 0$ for $i \neq j$. Then, the basis $|a\rangle$ are identified as the preferred representation for computing Born statistics.

how to select the basis in the context where nothing is assumed other than unitary QM, applied to the whole universe. The emergence of the very stable semi-autonomous environments in which subsystems are distinctly identifiable *is* the problem. As we will show, this will be addressed in our model. Furthermore, it can be demonstrated that, irrespective of the chosen basis, a quantum system coupled to an environment possessing larger number of degrees of freedom will generally exhibit waning interference [24]. This result undermines the argument for "ein"-selection from decoherence. Even if decoherence provides *a* mechanism for basis selection in specific cases, it does not do so uniquely. This leaves the original problem unresolved.

Although this problem arises in a general unitary QM, to make it more specific and tailored in the case of the QFT framework, it should be understood in connection with the basis-representation of the Fock space. Shall we describe Fock space in momentum space, or position space? Both are used in QFT models in computing physical quantities., but without a clear explanation as to what makes this determination. In particle physics and computation of scattering amplitudes, momentum representation is invoked to model incoming and outgoing particles with known energy and momentum. On the other hand, in condensed matter problems, usually a field representation is used, in the manner consistent with statistical field theory. We will put forward an answer to this problem in Section. 2.4.

2.2.3 The interpretation problem

Is QFT about particles, fields, or something else entirely? It is not clear whether it is possible to interpret QFT in a specific general way throughout. Depending on the model, different particle or field related aspects are highlighted, with no consistent unifying picture [25].

The particle ontology is primarily highlighted, especially in light of the Standard Model. This interpretation naturally originates from relativistic QM, which is one routes to QFT. As shown by Dirac in 1932 [26], relativity and QM imply creation and annihilation of states. This paves the way for the interpretation in terms of particles that are not conserved under interaction. This requires construction of Fock space as a symmetric/Grassman algbera, on which Weyl/Clifford associative algebra acts, corresponding to commutating/anti-commutating algebra of creation and annihilation of bosonic/fermionic particles. However, the construction of Fock space and canonical (anti-)commutation relations cannot be taken for granted in any QFT model. On flat space, one can always construct this structure for the free model, and treat any interacting model perturbatively on the same space. However, as AQFT has

demonstrated, in a general interacting theory this is not possible [11].

More generally, one can see the breakdown of particle interpretation more simply through the vacuum state. Under a non-inertial frame transformation, due to Unruh effect, the particle number fails to be conserved. This makes the very notion of "vacuum" frame-dependent, and not a universal aspect as required by an exclusively "particle" interpretation. Moreover, we know that the Fock space vacuum is not a physical vacuum at all, as it is filled with quantum fluctuations. These very fluctuations determine the stability of the vacuum state, and hence, have real physical consequences. It is not clear how this phenomenon fits in coherently with the particle interpretation of QFT where vacuum is a state where there is no particle, and hence, *no* physical entity by definition.

The plethora of interpretive challenges surrounding particles leads to an alternative perspective in QFT, framed in terms of fields. Let us define the field configuration basis as eigenvectors of field operators,

$$\hat{\phi}_x \left|\Phi\right\rangle = \phi_x \left|\Phi\right\rangle . \tag{2.10}$$

In this picture, vacuum state $|0\rangle$ is seen as a minimal-fluctuation static Gaussian, as one can show [8]

$$\langle\Phi|0\rangle = \mathcal{N} e^{-\frac{1}{2}\int_{x,y}\omega_{x,y}\phi_x\phi_y} . \tag{2.11}$$

Here, $\mathcal{N}$ is a normalization factor, and $\omega_{x,y}$ is the square root of the integral kernel in the potential. If we assume the free field Hamiltonian

$$H = \int_x \frac{1}{2}\pi_x^2 + \frac{1}{2}\phi_x(m^2 - \Delta)\phi_x, \tag{2.12}$$

where π_x is the conjugate momentum to ϕ_x, and m denotes the mass, then it follows

$$\omega_{x,y} := \sqrt{(m^2 - \Delta)}(x,y) = \int_p \frac{1}{(2\pi)^3}\sqrt{m^2 + p^2}\, e^{-ip\cdot(x-y)} . \tag{2.13}$$

One can readily verify Eq. (2.11). We can quantize Eq. (2.12) in field basis where

$$\langle\Phi|\,\hat{\pi}_x = -i\frac{\delta}{\delta\phi_x}\langle\Phi| . \tag{2.14}$$

The action of the Hamiltonian on the vacuum state can be computed as

$$\begin{aligned}\langle\Phi|\,\hat{H}\,|0\rangle &= \langle\Phi|\int_x\left(\frac{1}{2}\hat{\pi}_x^2+\frac{1}{2}\hat{\phi}_x(m^2-\Delta)\hat{\phi}_x\right)|0\rangle\\ &= \mathcal{N}\int_x\left(-\frac{1}{2}\frac{\delta^2}{\delta\phi_x^2}+\frac{1}{2}\phi_x(m^2-\Delta)\phi_x\right)e^{-\frac{1}{2}\int_{x,y}\omega_{x,y}\phi_x\phi_y}\\ &= \frac{1}{2}\mathcal{N}\int_x\omega_{x,x}\,e^{-\frac{1}{2}\int_{x,y}\omega_{x,y}\phi(x)\phi(y)} = \frac{1}{2}\int_x\omega_{x,x}\langle\Phi|0\rangle,\end{aligned} \tag{2.15}$$

which is the eigenstate with the ground energy $\int_p \frac{1}{2}\sqrt{p^2+m^2}$ as the eigenvalue. Also, it should be noted that the Gaussian state has minimal fluctuation as it minimizes the uncertainties bound by uncertainty relations. This is another way to see its equivalence with the ground state.

The state described in Eq. (2.11) illustrates that the term "vacuum" is in fact a misnomer, as it is better understood as a state with fluctuating fields centered around zero. Thus, the field interpretation offers a clearer understanding of the vacuum. Additionally, this interpretation has the merit of providing a mental picture of physical phenomena related to vacuum very nicely, such as the Casimir force and fluctuations.

On the other hand, it is not clear how to link this to scattering problems in particle physics. It is mathematically feasible to write a state with definite momenta $|p\rangle$ as a superposition of field configurations, as

$$\begin{aligned}|p\rangle &= \int_x\frac{1}{(2\pi)^3}e^{-ip\cdot x}\hat{\phi}_x^\dagger\,|0\rangle\\ &= \int D[\phi]\int_x\frac{1}{(2\pi)^3}e^{-ip\cdot x}\phi_x^*\,|\Phi\rangle\,\langle\Phi|0\rangle\\ &= \mathcal{N}\int D[\phi]\int_x\frac{1}{(2\pi)^3}e^{-\frac{1}{2}\int_{x,y}\omega_{x,y}\phi_x\phi_y-ip\cdot x}\phi_x^*\,|\Phi\rangle\,,\end{aligned} \tag{2.16}$$

which is rewritten in terms of field basis by the use of Eq. (2.11). But it is not clear how to associates the field interpretation with the said phenomenon. After all, we do not prepare incoming states in specific field configurations. In scattering problems, the empirical information and the interpretation thereof is about particles.

The interpretation problem already exists in QM. It is hidden behind wave-particle duality. Although to our disagreement, Bohr's Principle of Complementarity [27] essentially tried to make a cogent sense of this issue by undermining the necessity for having a unified interpretive picture in the first place. However, we insist upon such need and thus, acknowledge this problem. In QFT framework, one might say this issue expands even further into what may be called wave-particle-*field* triality. The reason is

that in addition to the general wave aspect arising from the interference within wave function in all QM models, both the particle and the field aspects are also highlighted separately, depending on both the model we build and the empirical data we collect.

2.2.4 The ultraviolet divergence problem

Perhaps the most well-known, most studied, oldest, and technical problem of QFT is its notorious UV divergence appearing in each mathematical formulation of.

The standard knowledge is that renormalization solves this problem. This is, however, misleading, if not totally false. One must distinguish *regularization* from *renormalization*.

Regularization refers to a mathematical technique used to rearrange or manipulate divergent sums, or more generally ill-defined problems, in order to obtain a finite value. In physical contexts, such as quantum field theory (QFT), regularization involves removing divergent contributions through specific methods (e.g., dynamical regularization, Pauli–Villars regularization, or cutoffs), thereby isolating the finite contribution that would otherwise be masked by infinities.

Let us formalize this. A QFT model is equipped with a Schrödinger equation of the form

$$i\partial_t \Psi = \hat{H}\Psi \tag{2.17}$$

for the wave functional Ψ. Let us assume the initial Ψ is normalizable with respect to a functional measure, for example,

$$D[\phi] = \prod_x d\phi_x, \tag{2.18}$$

for configurations ϕ_x. For Eq. (2.17) to be well-defined, the issue of regularization naturally arises. In local theories on configuration spaces with continuum distributions, this equation typically develops divergences during evolution. In other words, the propagator $\hat{U} = e^{-i\hat{H}}$ fails to exist. This failure in dynamical preservation of normalized wave-functions due to divergent terms characterizes a non-regularized QFT.

The reason stems from the local point-coincident kinetic term in the Hamiltonian, that is,

$$\hat{T} = \frac{1}{2}\int_x \hat{\pi}_x \hat{\pi}_x, \tag{2.19}$$

where π is the conjugate momentum to the fields. Eq. (2.19) represents a point-coincident coupling of π to itself, which, when defined over a non-separable Hilbert space consisting of continuum fields that can vary on arbitrarily small scales, diverges. One remedy is to regularize the kinetic metric, e.g.,

$$\hat{T} \to \frac{1}{2} \int_{x,y} \hat{\pi}_x g_{x,y} \hat{\pi}_x, \tag{2.20}$$

where $g_{x,y}$ is a localized distribution over a compact region in neighborhood of $x = y$ beyond which it falls off. Alternatively, one could truncate the fields at some small scale and define a configuration space of fields with a short-distance cutoff. By introducing a short-distance cutoff, we obtain a regularized Hamiltonian $\hat{H}^\epsilon$ that depends on the cutoff ϵ. This cutoff modifies the space of configurations. For any non-zero ϵ, only a subset of configurations on $\mathcal{S}$ satisfies the regularization condition and continuum fields correspond to $\epsilon = 0$.

Regularization may thus be viewed as a mapping that assigns to each configuration a regularized image in which short-distance features are suppressed or truncated, depending on the chosen scheme. The cutoff parameter is taken from the set of positive real numbers. Accordingly, we define the regularization function

$$\mathcal{R} : \mathcal{Q} \times \mathbb{R}_{>0} \longrightarrow \mathcal{Q}. \tag{2.21}$$

$\mathcal{R}(\phi, \epsilon) = \phi^\epsilon$ denotes the regularized image of a given configuration $\phi \in \mathcal{Q}$.

Regularizing the configurations, and consequently, the functional measure and inner product on the Hilbert space, renders the quantum theory well-defined by eliminating divergences.

The renormalization program, started already by Dirac in context of charge renormalization in early 1930s [28], and perfected through Wilson-Kadanoff approach[5] [30–32], explains how the accessible low-energy physics becomes independent of the high-energy physics. Therefore, *regardless* of any UV divergence which can be mathematically explained away by placing a cutoff as regulator, one can do effective physics. But in reality, this trick is bypassing the problem, not solving it directly. The question of what happens in the fine regime (or equivalently, high energy modes) remains open in the general case. This problem is usually considered to be part of the long-sought-after high-energy physics.

In fact, more broadly, the essence of renormalization extends beyond the framework

[5]It is noteworthy that the Wilsonian approach was already put forward by Stueckelberg and his student, Petermann in 1952, almost two decades before [29]. But it did not gain recognition historically due to the manner of presentation.

of QFT. At its core, this procedure reflects the recognition of the scale dependence of effective physics: depending on the scale of interest in studying a phenomenon, the physical description must be readjusted accordingly. A theory can thus be analyzed from this perspective independently of any technical problem of divergences. This viewpoint is standard in statistical models, where different phases and transition points are characterized by the behavior of the RG flow. Although less common, the concepts behind renormalization can also be applied to dynamical N-body models [33].

2.3 Principles of Pure Shape Dynamics

We recapitulate the Principles of PSD [2] below. The most relevant way to formulate the principles is to separate them into spatial and temporal aspects.

PSD demands shapes, and only shape variables. Depending on the model, shapes should be defined as scale-invariant relational configurations. For instance, if a model is discrete and consists of a set of points in a Riemannian manifold, the ratios of the inter-point separations constitute shape variables. The set of these variables define a compact Riemannian manifold, called shape space. Formally, the configuration space of N points in an open 3D Riemannian space is $\mathbb{R}^{3N}$. We are assuming the space is maximally symmetric to render the position and orientation of points irrelevant, and only their inter-relations as relevant DoFs. Under a suitable choice of the action of translational and rotational groups $T_3, \mathrm{SO}(3)$, one can construct the relational configuration space $Q_{\mathcal{R}} = \mathbb{R}^{3N}/T_3 \rtimes \mathrm{SO}(3)$.

The elements of Q_R comprise of separations between points, as well as the intrinsic left/right-handedness of the point configurations, which are not taken away by quotienting w.r.t rotations. Therefore, to construct shape space, we do a further quotient to shape space $\mathcal{S} = Q_R/P \times \mathbb{R}^+$. Shape space inherits the original kinetic metric on $\mathbb{R}^{3N}$ through best-matching procedure [34]. Let us assume the standard kinetic term is given as

$$T = g^{IJ}_{ab}\dot{q}^a_I \dot{q}^b_J, \tag{2.22}$$

where in the case of point-particles with masses in a Riemannian space with metric g^R_{ab}, it is explicitly $\tilde{g}^{IJ}_{ab} = \frac{1}{2}m_I\delta^{IJ}g^R_{ab}$. $I, J = 1, 2, \cdots N$ denote the particle indices. The shape space metric follows by minimizing the kinetic term along the action of Lie algebra of the similarity group $\mathrm{Lie}[\mathrm{Sim}(3)] = t_3 \oplus \mathrm{so}(3) \oplus r^+$. This means that shape

metric is

$$ds^2 = g_{ab}\delta q_s^a \delta q_s^b = \min_{\mathfrak{g}\in \text{Lie}[\text{Sim}(3)]} g_{ab}^{IJ}\, \mathfrak{g}[\delta q_I^a]\mathfrak{g}[\delta q_J^b] \tag{2.23}$$

The explicit expression for the algebra and construction of the metric are presented in Appendix C.1, C.2.

It is asserted by PSD as first principle that any physical model should be written as a autonomous dynamical system on $\mathcal{S}$, using the geometric structures on $\mathcal{S}$ alone. This means no external time or measure of scale should be assumed. Assuming shape coordinates q^a on $\mathcal{S}$, the dynamical system should have the general form

$$\begin{array}{rcl} \frac{dq^a}{ds} & = & u^a(q,\phi_A), \\ \frac{d\phi_A}{ds} & = & \Xi_1(q,\phi_A,\kappa,\cdots), \\ \frac{d\kappa}{ds} & = & \Xi_2(q,\phi_A,\kappa,\cdots), \\ \cdots & & \end{array} \tag{2.24}$$

ds denotes the arc-length with respect to the shape space metric. We can further generalize the PSD description in Eq. (2.24) by considering the ratios of the change in shape variables to dispense with the arbitrary parametrization. These quantities can be constructed as $\frac{d\phi_A}{ds}/\frac{dq^a}{ds} = \frac{d\phi_A}{dq^a}$, $\frac{dq^a}{ds}/\frac{d\kappa}{ds} = \frac{dq^a}{d\kappa}$, etc. Therefore, the dynamical variables are the shape variables, the directions on unit-tangent bundle, unit-acceleration bundle, as well as higher-order tangent bundles, all constructed from unparametrized curves on shape space and evolving with respect to one another. The complete construction is carried out in [2]. Thus, we formulate the principle as

Principle of PSD I:

> The relational dynamics of the universe must form an autonomous dynamical system on $\mathcal{S}$, using only the intrinsic geometric structures, the metric, and orders of scaleless tangent bundles constructed for unparametrized smooth curves γ on $\mathcal{S}$.

This principle addresses the permissible spatio-temporal structures for PSD models. With regard to duration of time, given only unparametrized curves are allowed, any parameter, such as the one used in Eq. (2.24) should be constructed from shape structures. However, this does not address all aspects to time. Leaving nuanced philosophical discussions aside, there are two important physical aspects to any model of time: duration and direction. *Principle I* demands duration to be solely dependent on shape evolution. However, we need an additional principle.

It was one of the triumphs of SD program that the SD model of Newtonian gravity, once constructed and cast onto shape space, leads to a dynamically generated arrow of time [35]. This means that although Newtonian gravity constructed with respect to an external inertial frame with spatio-temporal references exhibits time-reversal symmetry, once scale and external time are stripped away, there is an unambiguous arrow on shape space which can be identified with the arrow of time.

In more details, the solution lies in the analogue of Newton's potential on shape space. Newton's potential is scale-dependent, and has the homogeneity degree of -1. Once made scale-invariant by the total scale, it defines the "shape potential". The negative of shape potential is a remarkable quantity which was named "complexity" in [35].

For the model of N points with masses on Euclidean space, complexity is explicitly

$$C = \sqrt{\sum_{i,j} m_i m_j |r_i - r_j|^2} \sum_{i,j} \frac{m_i m_j}{|r_i - r_j|}, \tag{2.25}$$

where $|r_i - r_j|$ denotes the inter-particle distances, and m_i's are the masses. C is a positive-definite scale invariant function of only separations. This means that the total scale of the system is irrelevant, as the expression Eq. (2.25) is invariant under a global rescaling in the form $r_i \to \alpha r_i$. In fact, the purpose of using the square root of moment of center-of-mass inertia,

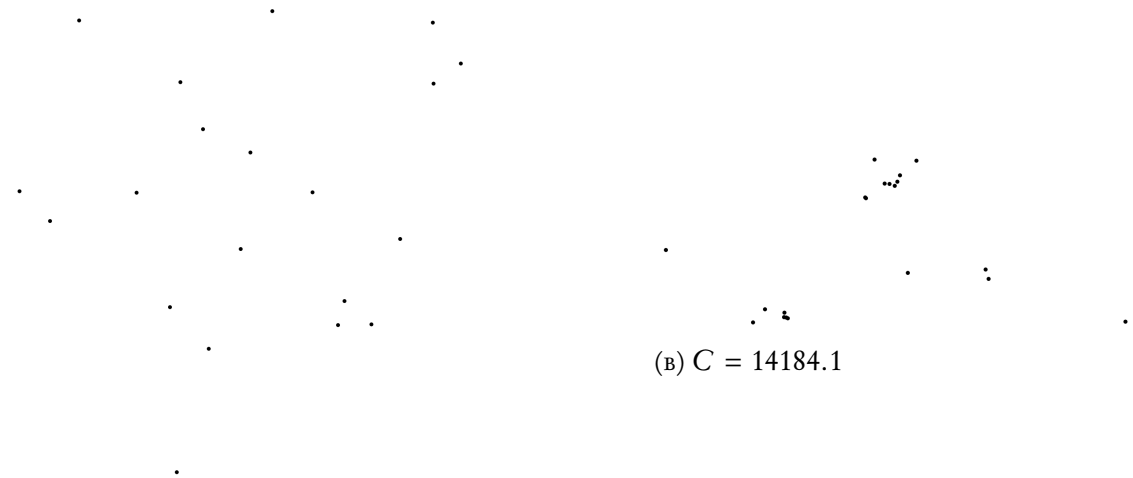
$$I_{\mathrm{CM}} = \sum_{i,j} m_i m_j |r_i - r_j|^2, \tag{2.26}$$

as a measure of scale, is to render the scale-dependent Newton's potential scale-invariant.

Complexity admits a minimum, and increases to infinity as points become more and more clustered. In fact, the best interpretation of complexity is a measure of *clustering* within a configuration.

The proposal for the arrow of time rests on behavior of C as a function of dynamical curves on $\mathcal{S}$. Complexity introduces an asymmetrical structure on shape space, with a number of singularities. All SD solutions exhibit a secular growth of complexity asymptotically, in all directions. In fact, it can be generally proven that shape dynamics is attractor-driven on shape space, subject to singularities of C as limit points. Figure 2.1 shows a typical example. A randomly generated distribution with low complexity dynamically evolves into a clustered high-complexity distribution.

In order to formulate a general principle as motivated by this finding, which will also be applicable to our work below, we need a PSD principle for the direction of



(B) $C = 14184.1$

(A) $C = 3986.06$

FIGURE 2.1: A randomly generated 2D configuration (A) with 20 points, sampled uniformly from a box in Euclidean space, dynamically evolves into configuration (B) with increased complexity. The clustering of the distribution is evident.

time. We turn the standard story upside down, and *demand* there to be a function on shape space, whose *monotonic* (and not just secular) growth defines an arrow of time.

Principle of PSD II/Principle of Monotonic Increase (PMI):
There must exist an at least C^1-smooth function on $\mathcal{S}$ that changes monotonically along the solution curves on $\mathcal{S}$. The arrow of time within any unparametrized solution curve is then identified with the direction of growth of the function.

This principle generalizes the observation of the gravitational arrows in shape dynamics. What concerns us here is a general PSD principle for the direction of time. Identifying time with a structure on shape space leads to remarkable results, as elaborated below.

2.4 THE CONCEPTUAL LAYOUT OF PSD APPROACH TO QFT

We present a blueprint for applying the principles of Pure Shape Dynamics (PSD), as stated in Section 2.3, to establish the foundation of a Pure Shape Dynamics Quantum Field Theory framework. In particular, we will discuss how these principles provide a coherent remedy for each of the four conceptual problems outlined in Section 2.2.

As previously discussed, the overall strategy of PSD is to identify and construct the shape space as the preferred kinematical state space, and to formulate a closed geometric dynamical system within it. Accordingly, the first step is to propose a

suitable candidate for the shape structure corresponding to a given Quantum Field Theory (QFT) model.

As will be explored in both the fermionic and bosonic models in Chapters 3 and 4, a natural extension of the notion of shape space from the N-body setting is the concept of charge or fermionic density. The density distribution of charged fermions has the advantage of encoding spatial structures. One can then define the configuration space as the space of such distributions satisfying certain boundary conditions.[6]

As the next step, one should separate the elements of the configuration that are induced by background structures and construct the corresponding shape space. This can be achieved by defining the density function on a Riemannian space and then imposing diffeomorphism and conformal constraints. These constraints ensure that only the intrinsic structure within the density is regarded as physical, rather than its embedding within the background.

If we formulate a field Schrödinger equation on a shape space constructed in this manner, we can model the dynamics of the fermionic or charge density—provided that the dynamics remain well-defined, that is, regularized. As explained below, the implementation of Principle PSD II addresses this requirement. The formal construction will be developed in the subsequent chapters. Let us now examine how this overall strategy is expected to resolve the aforementioned problems.

With regard to the measurement (reality) problem, the approach advocated here is to invoke the de Broglie–Bohm (dBB) model [17, 36]. A concise introduction to this formulation is provided in Appendix A. The dBB model supplements the Schrödinger equation with a dynamical system for a single realized configuration, referred to as the dBB (or often Bohmian) configuration. The dynamics of this configuration is governed by the "guidance equation", named after de Broglie's original formulation [37]. David Bohm demonstrated that such a dynamical system reproduces the statistical predictions of quantum mechanics [17]. Moreover, by introducing an objective micro-process associated with the dynamics of the dBB configuration, the model eliminates the measurement problem.

The adoption of the dBB model therefore resolves the first, and arguably the most serious, conceptual issue. Furthermore, the dBB framework is generally compatible only with the configuration-space representation (see the discussion at the end of Section A.1). It thus resolves the representational ambiguity in quantum systems by identifying the configuration basis as the fundamental one.

[6] From the PSD standpoint, we generally remain silent regarding the choice of boundary conditions, allowing different models to impose different ones. But we state that either compact spaces with periodic boundary conditions or non-compact spaces with asymptotically vanishing distributions are admissible.

Even regardless of the use of the dBB model, our conceptualization of shapes in *Principle I* as scale-invariant structures of interrelations within a distribution of charged particles already selects a particular representation. A Quantum Field Theory model must therefore be formulated in terms of the position representation of fields in order to define the relevant distributions and their corresponding shape descriptions.

We also argue that the empirical data we obtain, including those of Quantum Field Theory, are registered and recorded within *spatially* localized subsystems. For instance, computing scattering cross-sections requires an experimental setup in which an inflow of particles interacts locally before the scattered products are observed across space. Likewise, in condensed matter systems, the observables pertain to interacting material structures. Hence, we contend that the shape representation is, in principle, capable of reproducing empirical data.

In summary, both the dBB model and *Principle I* of Pure Shape Dynamics address the representation problem, with the latter providing stronger grounds.

The shape representation also provides a unified interpretation of quantum fields as a bookkeeping device for registering the distribution of charged fermionic particles. Since the very notion of shape requires considering distributions *universally*, this interpretation avoids the difficulties associated with the standard particle interpretation. Even when the quantum fields are in the vacuum state, there must exist bound structures in the environment to enable the detection and registration of fluctuations. The Unruh effect, for instance, necessarily relies on constructing a non-inertial observer equipped with a reference system. Such reference systems are themselves part of the global shape of the universe and, therefore, part of the structure represented by quantum fields. By treating the total distribution in the universe as fundamental, the conceptual issues surrounding the "vacuum", which is typically understood as a state with no particle excitations, disappear.

Thus far, *Principle I* of PSD has been employed to address the representation and interpretation problems of QFT. The fourth issue, namely that of regularizability, can be addressed through *Principle II*. The key idea, first suggested by Julian Barbour,[7] arises from the intuitive observation that the complexity function Eq. (2.25) is highly sensitive to small-scale separations. That is, if for instance, if there are two particles i^*, j^* much closer to each other than the rest of the inter-particle separations, Eq. (2.25)

[7]This was communicated in private.

can be approximated as

$$C \approx \frac{\sqrt{\sum_{i,j} m_i m_j |r_i - r_j|^2}}{|r_{i^*} - r_{j^*}|} + O(|r_{i^*} - r_{j^*}|), \tag{2.27}$$

In the limit that $|r_{i^*} - r_{j^*}| \to 0$, complexity diverges.

This simple observation can be directly related to the regularizability of QFT models. One way to understand the divergences in QFT is through the lens of lattice quantization. In the continuum limit, as the small-scale distances between lattice sites approach zero, the evolution kernel, which integrates over all possible modes, diverges. By keeping the small-scale distances finite, one effectively introduces a cutoff, thereby ensuring a well-defined evolution. This is, in essence, the principle underlying lattice regularization, as will be more explicitly demonstrated in Chapter 3.

The behavior of complexity is consistent with this interpretation and, consequently, provides a quantitative model for the emergence of divergences as a consequence of the continuum structure. As long as the complexity of a particle distribution remains finite, there exists a non-zero short-distance scale, roughly proportional to the inverse of the complexity. Conceptually, this implies that, under a suitably generalized definition of complexity for quantum fields, the theory should be *naturally* regularized so long as the complexity remains finite. Given the well-understood relationship between complexity and the conventional high-momentum cutoff used in regularization, complexity can in fact be regarded as the natural regulator.

The significance of *Principle II* lies in ensuring that a suitable definition of complexity exists, remains finite throughout evolution, and grows monotonically. If one formulates such a function that is sensitive to short-distance features within a distribution, the interpretation of complexity as a regulator follows naturally. By maintaining the finiteness of this function, configurations are regularized by construction. Consequently, one obtains a regularized QFT model with a state-dependent cutoff. Furthermore, because *Principle II* requires the function to grow during evolution, the cutoff itself increases dynamically, allowing higher-energy modes to progressively become active.

We shall formalize this conceptual picture. Assume that the shape space $\mathcal{S}$, corresponding to configurations of distributions, exists. Here, the application of *Principle II* to QFT consists in identifying a complexity functional C on $\mathcal{S}$ whose value is monotonically related to the short-distance cutoff imposed on configurations, in the

manner described in Eq. (2.21). Accordingly, for any configuration ϕ, we require

$$\frac{dC}{d\epsilon^{-1}}[\phi] = \frac{dC[\mathcal{R}(\phi,\epsilon)]}{d\epsilon^{-1}} > 0. \tag{2.28}$$

In particular, we also demand that

$$\lim_{\epsilon\to 0} C[\mathcal{R}(\phi,\epsilon)] = \infty, \tag{2.29}$$

as evaluated for any configuration in $\mathcal{S}$.

There are, in general, multiple ways to incorporate the complexity cutoff into the dynamical equation to render the quantum model "complexity-regularized". We will explore these approaches in the subsequent concrete models. In general, let us express the modified dynamics in the form

$$\frac{d}{dt}\Psi = \mathcal{H}[\Psi, C], \tag{2.30}$$

where $\hat{H}\Psi$ has been replaced by a more general functional of the wave function and the complexity cutoff. Eq. (2.30) may or may not remain linear, which remains as a point that shall be examined in Section 3.4. Moreover, the complexity of the entire system is assumed to follow its own monotonic dynamics, namely,

$$\frac{dC}{dt} > 0, \tag{2.31}$$

in keeping with *Principle II*.

Once the complexity regulator is implemented, the left-hand side of Eq. (2.30) can be rewritten with respect to complexity itself as the temporal parameter:

$$\frac{d}{dC}\Psi = \mathcal{H}'[\Psi, C], \tag{2.32}$$

which fully realizes the identification of complexity with time, as prescribed by the PSD principles.

In summary, Eq. (2.30) describes a regularized QFT evolution in which C acts simultaneously as both the evolution parameter and the cutoff. This is the essence of *Principle II/PMI*.

Chapter 3

The Fermionic Density Model

"The distribution of fermion number in the world certainly includes the positions of instruments, instrument pointers, ink on paper, ... and much much more."

John Bell [38]

3.1 Motivation from Quantum Gravity

In this chapter, a fermionic QFT model is constructed in a manner consistent with the PSD-inspired conceptual foundation outlined in Section 2.4. This particular model, aside from conceptual reasons, is motivated by expectations from Quantum Gravity, ensuring that the following investigation remains physically significant.

Let us the low-energy limit of a general Quantum Gravity model which has yet to be constructed, but can be sketched as follows. Consider the classical ADM Hamiltonian for GR in triad formulation (see Appendix B)

$$\mathcal{H}_g[e,\pi] = \int_x N_0\, S_g = \frac{1}{16\pi G}\int_x N_0 \left[\frac{1}{2|e|}\pi_i^a\left(e_a^j e_b^i - \frac{1}{2}e_a^i e_b^j\right)\pi_j^b - (R[e]-2\Lambda)|e|\right] \tag{3.1}$$

along with the matter sector

$$\mathcal{H}_m[\psi^*,\psi,e] = \int_x N_0\, S_m = \int_x N_0\,|e|\,\psi_\alpha^*\left[ie_i^a(\gamma^0\gamma^i)_\beta^\alpha\left(\delta_\sigma^\beta\partial_a + \frac{1}{4}\omega_a^{JK}(\gamma_{JK})_\sigma^\beta\right) + m\,(\gamma^0)_\sigma^\alpha\right]\psi^\sigma, \tag{3.2}$$

where π_i^a denotes the canonically conjugate momentum density corresponding to the

spatial triad e^i_a with the modulus of determinant $|e|$, such that the spatial metric is $g_{ab} = e^i_a e^j_b \delta_{ij}$. $R[e]$ is the spatial Ricci scalar calculated from the spatial triad and Λ the cosmological constant. N_0 denotes the homothetic (shape[1]-preserving) lapse which decouples the scale factor from shape variables, derived in [6]. It means, using N_0,

$$\frac{d}{dt}\sqrt{g}_x \equiv \{\sqrt{g}_x, N_0 \cdot S_g\} = c\sqrt{g}_x \tag{3.3}$$

for a constant c. The evolution of scale, $\sqrt{g}$, does not depend on the shape components of metric and remains dynamically closed, hence, the choice of lapse has rendered its evolution homothetic. The condition Eq. (3.3) can be used to explicitly solve for N_0 algebraically. As for the matter Hamiltonian, ψ^α is a Dirac spinor field, γ^I the Dirac matrices , $\gamma^{IJ} = \frac{1}{2}[\gamma^I, \gamma^J]$ and ω^{JK}_a the spatial spin connection compatible with e^i_a. We keep using natural units where $c = 1, \hbar = 1$. Thus, the only dimensionful quantities and their relations are

$$[m]^{-1} = [\psi]^{-2/3} = [\Lambda]^{-2} = [G]^{1/2} = [|e|]^{1/3} = [\text{Length}]. \tag{3.4}$$

Finally, the Hamiltonian constraint

$$\frac{1}{16\pi G}\mathcal{S}_g[e,\pi] + \mathcal{S}_m[\psi^*, \psi, e] \approx 0 \tag{3.5}$$

and momentum constraint

$$P[e,\pi,\psi^*,\psi,\xi] = \int_x \left(\pi^a_i \mathcal{L}_\xi e^i_a + i\,\xi^a \psi^*_\alpha \left((\gamma^0)^\alpha_\beta \partial_a + \frac{1}{4}\omega^{JK}_a (\gamma^o \gamma_{JK})^\alpha_\beta\right)\psi^\beta\right) \approx 0 \quad \forall \xi \tag{3.6}$$

have to be imposed on the initial-value data. ξ^a denotes the shift vector field which generates spatial diffeomorphism transformations. $\mathcal{L}_\xi$ denotes the Lie derivative.

As touched upon, this classical gravity model, by construction and after implementing the lapse function N_o, decouples the evolution of the scale factor $\omega := |g|^{\frac{1}{2}} = |e|$ from the scale-invariant part of the metric, that is, the shape variables $\theta^i_a = e^i_a/|e|$. Let us denote the triad of shape momenta as η^a_i, defined to be conjugate to e^i_a. Hence, we have a dynamical system for shape parts alone, satisfying the constraints Eqs. (3.5) and (3.6), as constructed in [6].

It is of value to present the geometric sector of the Hamiltonian Eq. (3.1) explicitly

[1] In the context of dynamical geometry, shape refers to the metric up to diffeomorphisms and conformal transformations. See [6] for the details.

as functional of the conjugate shape triads e^i_a, η^a_i and scale factor ω:

$$S_g = \frac{1}{16\pi G}\left[\frac{1}{2|e|}\pi^a_i\left(e^j_a e^i_b - \frac{1}{2}e^i_a e^j_b\right)\pi^b_j - (R[e] - 2\Lambda)|e|\right]. \tag{3.7}$$

Furthermore, the homothetic lapse function that solves Eq. (3.3) is

$$N_0 = \frac{2c}{\sqrt{6\left|\frac{\eta^a_i e^j_a e^i_b \eta^b_j}{\omega} - R[\omega^{\frac{1}{3}} e^a_i] + 2\Lambda + S_m\right|}}. \tag{3.8}$$

This theory describes a simple fermionic model with Yukawa interaction, i.e., the term

$$\psi^\dagger \gamma^0 \psi \phi \tag{3.9}$$

resulting from $\mathcal{H}_g + \mathcal{H}_m$ in the non-relativistic weak-gravity regime. But let us sketch the steps we believe argue how full Quantum Gravity leads to our fermionic model:

Step 1:

We assume that a well-defined complexity-regularized Quantum Gravity model exists with an evolution with respect to a complexity functional, as outlined in Chapter 2, Section 2.4. As an effective description, we assume the wave functional

$$\Psi[\theta, \psi] \tag{3.10}$$

exists in the Hilbert space on conformal superspace (shape space of dynamical geometry), and evolves according to a Schrödinger equation with effective Hamiltonian

$$\hat{H}_{\text{eff}} := (\mathcal{H}_g + \mathcal{H}_m)[\theta, \psi, \pi \to -i\frac{\delta}{\delta_\theta}, \psi^* \to -\frac{\delta}{\delta_\psi}], \tag{3.11}$$

where standard quantization has been applied. We note that the Hamiltonian at this stage is still dependent on the scale factor ω, as is apparent from Eq. (3.7). This leaves the dynamics open and dependent on the scale. Therefore, to close the Schrödinger equation for $\Psi[\theta, \psi]$, we impose the constraint such that it fixes the scale part:

$$\langle \Psi | \hat{\mathcal{S}}_g + \hat{\mathcal{S}}_m | \Psi \rangle [\omega] = 0, \tag{3.12}$$

where the quantized version of the bare Hamiltonian terms (with no N_0) are used. This expectation value is essentially the quantum analogue of the Hamiltonian constraint for the scale factor and is used to solve for ω for any given wave functional.

To view this important step from another angle, we emphasize that this approach circumvents the "problem of time" in Canonical Quantum Gravity [39] by imposing Eq. (3.12) as an expectation-value constraint, rather than as a operator constraint (see Appendix A.4). Therefore, this step here goes beyond the standard quantization technique, and opens up a promising possibility with new physics. The fact that it is possible to fix the Hamiltonian constraint in the scale sector and retain an evolving system for shapes has already been noted before [40], and it is one of the main motivations for shape space as the arena for Quantum Gravity.

Step 2:

We assume small quantum-gravitational fluctuations around the flat metric ($\theta^i_a = \delta^i_a + \delta\theta^i_a$). Therefore, we assume semi-classical approximation and write

$$\Psi = \Psi_o[\delta\theta]\Psi_m[\psi], \tag{3.13}$$

By our approximations, $\Psi_m[\psi]$ evolves under an effective Hamiltonian *independent* of geometric variables. For small θ on flat background, expanding Eq. (3.2) as $e^i_a = (1+\phi)\delta^i_a$ (keeping only scale factor to the first order) yields

$$\tilde{\mathcal{H}}_m[\psi^*,\psi] = \int_x \psi^*_\alpha \left[i(1+2\phi)(\gamma^0\gamma^i)^\alpha_\beta \partial_a + (1+\phi)m(\gamma^0)^\alpha_\beta \right] \psi^\beta. \tag{3.14}$$

The quantization of this Hamiltonian, $\hat{\tilde{H}}_{\text{eff}} := \tilde{\mathcal{H}}_m[\psi, \psi^* \to -\frac{\delta}{\delta\psi}]$ evolves $\Psi_m[\psi]$. This is the ground for Yukawa coupling. Finally, in this approximation, doing the expansions $e^i_a = \delta^i_a(1+\phi)$, $|e| = 1+3\phi$, and $R[e] = -4\Delta\phi$ up to the first order in ϕ, Eq. (3.12) reads

$$\langle \hat{S}_g + \hat{S}_m \rangle = 4\Delta\phi + 2\Lambda(1+3\phi) + 16\pi G \langle \Psi_m | \left((1+2\phi)\hat{H}^0_D - m\phi\hat{\psi}^\dagger\gamma^0\hat{\psi} \right) | \Psi_m \rangle = 0. \tag{3.15}$$

$\hat{H}^0_D$ denotes the Dirac Hamiltonian in flat background. Therefore, we have an effective Hamiltonian for the Dirac fields and the constraint Eq. (3.15) for the scale factor.

Step 3:

Finally, we shall take the non-relativistic limit to explore the low-energy slow limit of the QG model. All the terms in S_m can be ignored in favor of m, and setting $\Lambda = 0$, Eq. (3.15) yields

$$\Delta\phi = -16\pi G m \langle \psi^\dagger \gamma^0 \psi \rangle. \tag{3.16}$$

Therefore, under the assumption of small shape fluctuations of the geometry, the Dirac field only couples to the conformal mode of gravity—namely, the scale factor—which can be modeled as a Dirac QFT with Yukawa coupling. This shows how a simple fermionic model is expected to arise in the weak-gravity approximation of a fundamental PSDQG, a model which shall be explored in alignment with PSD below.

3.2 The setup

As explained in Chapter 2, fermionic distribution serves an ideal role as a preferred variable (ontology) for a quantum field model. It enjoys a clear interpretation as the building ingredient of the physical world, and thereby, fixes the representation ambiguity of quantum fields. Additionally, due to Pauli Exclusion Principle, it technically comes with the advantage of admitting less possible forms in the theory, as all terms involving $\psi\psi(x), \cdots$ vanish.

To formulate these fundamental degrees of freedom, consider Dirac quantum fields $\psi^\alpha(x)$ on the three-dimensional space where $\alpha = 1, 2, 3, 4$. Following Bell's similarly-motivated proposal [38], we define the fermion density operator[2] $\hat{N}(x) = \hat{\psi}^\dagger_\alpha(x)\hat{\psi}^\alpha(x)$ where Einstein summation convention is invoked for the spinor variables. Note that $\hat{N}(x)$ is defined without any normal-ordering. The total number density operator $\int_x \hat{N}(x)$ is essentially related to the conserved total charge in Dirac theory coupled to gauge fields. Let us assume the expansion

$$\hat{\psi}^\alpha_s(x) = \frac{1}{V} \sum_{p_a} \sqrt{\frac{m}{E_p}} \left(u^\alpha_s(p) \hat{c}_{p,s} e^{ip\cdot x} + v^\alpha_s(p) \hat{d}^\dagger_{p,s} e^{-ip\cdot x} \right), \tag{3.17}$$

where $u_s(p), v_s(p)$ are the eigenstates of the Dirac operator

$$D = -i\gamma^0\gamma^a\partial_a + m\gamma^0 \tag{3.18}$$

[2] Bell called it fermion "number" density operator, a terminology which can be misleading. This operator does not and cannot measure number of fermions, i.e., number of electrons and positrons. Therefore, we refer to it as simply fermion density to avoid confusion.

with positive and negative energies respectively, and $E_p = \sqrt{|p|^2 + m^2}$ is the relativistic energy. $\hat{c}_{p,s}, \hat{d}_{p,s}$ and their adjoint represent the creation/annihilation operators. We have chosen the dimensionless normalization condition

$$\begin{aligned} u_s^\dagger(p)u_r(p) &= \frac{E_p}{m}\delta_{rs}, \\ v_s^\dagger(p)v_r(p) &= \frac{E_p}{m}\delta_{rs}. \end{aligned} \tag{3.19}$$

Dirac 4×4 matrices satisfy Clifford algebra, i.e., $\gamma^0\gamma^0 = \pm 1$, $[\gamma^0, \gamma^a]_+ = 0$, $[\gamma^a, \gamma^b]_+ = \mp\delta^{ab}$. Moreover, the two corresponding sets of operators satisfy the only non-zero anti-commutation relations

$$\begin{aligned} [\hat{c}_{p,s}, \hat{c}^\dagger_{q,r}]_+ &= \delta_{sr}\delta_{pq}, \\ [\hat{d}_{p,s}, \hat{d}^\dagger_{q,r}]_+ &= \delta_{sr}\delta_{pq}. \end{aligned} \tag{3.20}$$

When $\hat{N}(x)$ acts on the states in Fock space, it counts the positive/negative-energy electronic modes created by $\hat{c}^\dagger_{p,s}$/ $\hat{d}_{q,r}$.[3] Therefore, it is even more accurate to call it electronic charge density operator.

As for our fundamental variable, in principle, we have two options which lead to two different strategies. We can treat the spectrum of $\hat{\psi}^\dagger_\alpha(x)\hat{\psi}^\alpha(x)$ as physically real, for which it is most natural to invoke de Broglie–Bohm (dBB) model. We would have to posit actual fermionic distributions whose value is picked from the eigenvalues of the fermion density operator, along with a guidance equation for its evolution. This approach is equivalent to the continuum limit of Bell's dBB fermionic quantum field theory model [41, 42]. Although there are many upsides to this, including complete resolution of the measurement problem as well as the other particular problems of QFT, we shall implement a second strategy which yields a more simple model within standard quantum mechanics.[4]

By taking the expectation value, let us define the density matrix as

$$\rho_{(x,s)(y,r)} := \langle \hat{\psi}^{r\dagger}_\alpha(y)\hat{\psi}^\alpha_s(x)\rangle. \tag{3.21}$$

This is the one-particle density matrix. To see that, assume a general wave functional

[3] From Dirac hole theory, $\hat{d}^\dagger_{q,r}$ is creation operator for a positron with momentum q and spin r, while $\hat{d}_{-q,-r}$ corresponds to the hole left by it in Dirac sea, that is, an electron with negative energy, $-E_q$.

[4] dBB will be employed in the next chapter in the bosonic field model.

is given:

$$|\Psi\rangle = \int_{x_1,x_2,\cdots} \Psi_{x_1,x_2,\cdots} |x_1, x_2, \cdots\rangle, \tag{3.22}$$

where states $|x_1, x_2, \cdots\rangle = \bigotimes |x_i\rangle$ are define by acting on the vacuum state, i.e., $\hat{\psi}^\dagger(x)|0\rangle$ (the spin indices have been dropped for simplicity). Then,

$$\langle\Psi|\hat{\psi}^\dagger_\alpha(y)\hat{\psi}^\alpha(x)|\Psi\rangle = \sum_i \int_{z_1,z_2,\cdots,z_{i-1},z_{i+1},\cdots} \Psi_{z_1,z_2,\cdots,z_{i-1},x,z_{i+1},\cdots}\Psi^*_{z_1,z_2,\cdots,z_{i-1},y,z_{i+1},\cdots}. \tag{3.23}$$

is the density matrix projected onto the one-particle sector of the Fock space, modulo a total normalization which can be fixed by imposing $\int_x \rho_{xx} = 1$.

We treat Eq. (3.47) as the fundamental variable for the fermionic model. The diagonal terms essentially define the charge distribution function. Some questions arise about its sufficiency. What information is exactly lost in transitioning from wave functional Ψ to ρ_{xy}, and to what extent can the latter explain observable phenomena?

We argue that so long as the model is construed as universal, $\rho_{x,y}$ suffices. That is because all the background structures, measuring apparatus and instruments, along with subsystems under study are part of the fermionic density-matrix distribution ρ_{xy}. All these subsystems consist of bound distributions of fermions which are in principle part of the diagonal terms, i.e., $\rho_{x,x}$. Therefore, this variable captures what is to universal formation of structures in both classical and quantum regimes. See Section **??** for the discussion.

To complete the model, we construct the dynamics. A simple as well as physically-motivated choice, as per our investigation of the weak non-relativistic Quantum Gravity, is Yukawa interaction between Dirac fields and a scalar ϕ. The Hamiltonian reads

$$\hat{H}[\bar{\psi}, \psi] = \int_x \hat{\psi}^\dagger D\hat{\psi} + g\hat{\bar{\psi}}\phi\hat{\psi}, \tag{3.24}$$

where $\hat{\bar{\psi}}(x) = \hat{\psi}^\dagger(x)\gamma^0$. To close the dynamics, we impose the elliptic constraint on the scalar at all times, assuming a foliation:

$$\Delta\phi_x = -\langle\hat{\bar{\psi}}\hat{\psi}\rangle(x), \tag{3.25}$$

where Δ is the Laplace-Beltrami operator. Therefore, we will have the equations of state from Eq. (3.24) along with condition Eq. (3.25) governing the dynamics of $\rho_{x,y}$. The equations will be derived in Section. 3.3.

3.3 Equations of state

We shall derive the evolution of the one-particle density matrix ρ_{ab}, where for simplicity, we have compacted both the configuration space index and spin ones into Latin index a. Leaving the regularization problem for next section, we assume we have a well-defined self-adjoint Hamiltonian $\hat{H}$, whose associated von Neumann evolution equation is

$$\dot{\rho}_{ab} = -i\langle[\hat{\psi}_b^\dagger \hat{\psi}^a, \hat{H}]\rangle. \tag{3.26}$$

Note that the evolution is given w.r.t some nominal time parameter implicit in $\hat{H}$. Given that *Principle II* of PSD will be implemented shortly, by virtue of which evolution parameter will be identified with complexity, we can keep this "Newtonian remnant" at this stage to be removed later.

To write down the equation more explicitly, let us assume the low-energy domain and take the non-relativistic limit. In this limit, where the support of positive-energy solutions screen off from the negative-energy ones, Eq. (3.26) reduces to the equation for only the positive ones, labeled by two spin modes. More rigorously, if we write

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{3.27}$$

then the eigenstate solutions $\psi(x,t) = e^{-iE_p t + ip\cdot x}\psi_p$ to Dirac equation

$$i\dot{\psi} = D\psi \tag{3.28}$$

for positive E_p read

$$\chi = \frac{\sigma \cdot p}{E_p + |p|}\phi \approx \frac{\sigma \cdot p}{2m}\phi + O(\frac{p^3}{m^3}) \tag{3.29}$$

in the non-relativistic limit. So, χ/ϕ is of order p/m and we can ignore χ in favor of ϕ, using the relation $\chi = \frac{-i\sigma\cdot\nabla}{2m}\phi$ back in the original equation. Also, given $\dot{\phi} \propto \phi^\dagger$, $\dot{\chi} \propto \chi^\dagger$, we shall keep only the terms involving $\phi^\dagger$ in the Hamiltonian to evolve the only relevant component of ψ in the non-relativisitc limit. Moreover, we redefine $\phi \to e^{imt}\phi$. Finally, the Hamiltonian in Eq. (3.24) becomes

$$\hat{H} = \int_x -\frac{1}{2m}\hat{\psi}_i^\dagger \Delta\hat{\psi}^i + g\phi\hat{\psi}_i^\dagger\hat{\psi}^i \equiv \hat{\psi}_a^\dagger A_b^a \hat{\psi}^b, \tag{3.30}$$

where $\psi_{1,2}$, denoting the redefined positive-energy component ϕ in Dirac spinor. The whole expression has been compactified into an operator, invoking Einstein summation/integration convention.

As for equations, it follows that

$$\begin{aligned}
[\hat{\psi}_b^\dagger \hat{\psi}^a, \hat{H}] &= [\hat{\psi}_b^\dagger \hat{\psi}^a, \hat{\psi}_c^\dagger A_d^c \hat{\psi}^d] \\
&= \hat{\psi}_b^\dagger [\hat{\psi}^a, \hat{\psi}_c^\dagger A_d^c \hat{\psi}^d] + [\hat{\psi}_b^\dagger, \hat{\psi}_c^\dagger A_d^c \hat{\psi}^d] \hat{\psi}^a \\
&= \hat{\psi}_b^\dagger \left([\hat{\psi}^a, \hat{\psi}_c^\dagger]_+ \hat{\psi}^d - \hat{\psi}_c^\dagger [\hat{\psi}^a, \hat{\psi}^d]_+ \right) A_d^c \\
&+ \left([\hat{\psi}_b^\dagger, \hat{\psi}_c^\dagger]_+ \hat{\psi}^d - \hat{\psi}_c^\dagger [\hat{\psi}_b^\dagger, \hat{\psi}^d]_+ \right) \hat{\psi}^a A_d^c \\
&= \hat{\psi}_b^\dagger A_d^a \hat{\psi}^d - \hat{\psi}_c^\dagger A_b^c \hat{\psi}^a .
\end{aligned} \tag{3.31}$$

So far the result is in basis-independent form. To use position basis in keeping with our principles,

$$\dot{\rho}_{xy} = -\frac{1}{2m} \left(\Delta_x \rho_{xy} - \Delta_y \rho_{xy} \right) + g(\phi_x - \phi_y) \rho_{xy} . \tag{3.32}$$

And the constraint

$$\Delta \phi_x = -\langle \hat{\psi}^\dagger(x) \hat{\psi}(x) \rangle \tag{3.33}$$

holds for ϕ_x.

Note the couplings and their dimensions as coming from Eqs. (3.1) and (3.2): In the original model, we have the mass m, the gravitational coupling G, and the total scale which has here been set to 1 nominally, while ϕ represents the local fluctuations in scale. By redefining ϕ and rendering it dimensionless[5], we have removed the coupling in the constraint Eq. (3.33). The total mass has also been made dimensionless along with the global scale. Also, a general dimensionless g in the dynamical part of the evolution is included in the totally dimensionless equation Eq. (3.32). This has simplified the equations without detracting anything from the physics, as the evolution of ρ matters in the end. However, in applying renormalization and studying the running of couplings, it will become relevant to restore the original couplings as one is interested in running of G and Λ (if present) primarily.

Let us pause and qualitatively analyze behavior of Eq. (3.32). The ultra-local interaction only manifests itself in dynamics of non-diagonal terms, creating interference within the density matrix. On the other hand, the distribution "density" part, i.e., ρ_{xx}, is affected by the first term through interference of different modes. So density

[5] To be exact, $\phi \to \frac{\phi}{4\pi G m}$.

changes at the second order through evolution of non-diagonal terms. As for the initial conditions, one needs to work out the quantum cosmological phase to make any precise statement. On physical grounds, we would expect some relatively uniform distributions comes out with seeds of minuscule inhomogeneities.

In standard QFT, Eq. (3.32) does not in fact exist, unless we regularize the kinetic term Δ with a high-momentum (equivalently, short-distance) cutoff. Otherwise, under time evolution, the dynamics becomes singular. This is the common problem with QFT models in continuum—namely, the regularization issue—as discussed in Chapter. 2.2.4. In the next section, we show how the implementation of our principle modifies this dynamics and renders the model *naturally* regularized.

3.4 Complexity functional as regulator

3.4.1 Implementation and general framework

The problem which regularization has inevitably to address is rendering the evolution in quantum fields well-defined. The issue essentially arises from considering all continuous field configurations in the measure, with no short-distance, or equivalently, high-momentum cutoff in the fourier representation. In the canonical formulation, this manifests in self-adjointness of $\hat{H}$ in the dynamics Eq. (3.26). In the path integral formulation, this is linked to divergent determinants in computing

$$F = \log \int D[\psi^*]D[\psi]e^{-\psi^\dagger A\psi + J^\dagger\psi + \psi^\dagger J} = \log \mathrm{Det}\, A + J^\dagger A^{-1} J, \tag{3.34}$$

where J is some source field. This quantity appears in computing the generating functional for connected n-point functions or free energy of the system, depending on perspective.[6] In the perturbative picture, the problem reduces to divergent loop-corrections that appear in expansion of $\mathrm{Det}\, A$.

The goal of our project is to identify an evolving *complexity functional* that not only gives rise to a universal arrow of time in keeping with the classical N-body model [35], but also defines a measure of how many short-distance modes there are in the system. As long as complexity grows, but remains finite, it then sets a bound to how *finely* the field distribution can vary, essentially setting a momentum cutoff. Implementation of the cutoff is the essential step in any regularization procedure. However, instead of treating it artificially as merely a mathematical technique, as in standard QFT, we posit that, in the fundamental model, there is a dynamical cutoff, given by a suitable

[6] See the discussion in Chapter. 2.1 for the two main perspectives on QFT models.

complexity functional of quantum fields.

Formally, let us denote such complexity functional by $C[\rho]$, whose value evolves as ρ does. Then, our *Principle II* from Section 2.4 demands the value of $C[\rho]$ at each instant regulates the quantum evolution which is otherwise ill-defined. To implement this, we shall invoke Principle of Monotonic Increase (PMI), enforcing C grows monotonically as a function of time, but remains finite, that is,

$$\frac{dC}{dt} > 0. \tag{3.35}$$

To implement Eq. (3.35) and thereby, regularize the model, the von Neumann equation Eq. (3.26) has to be modified. There is a general procedure for implementing PMI. We follow the same strategy tailored to the investigation here.

Let us write Eq. (3.26) in a form analogous to the geometric equations. Let A denote the general bi-indices, $A \equiv ab$. Define G_{AB} such that

$$\dot{\rho}_A = G_{AB}\rho_B, \tag{3.36}$$

where Einstein summation convention for repeated index is invoked. From Eq. (3.31), it follows that

$$G_{AB} \equiv G_{(ab)(cd)} = -i\left(\delta(b-d)H_{ac} - \delta(a-c)H_{db}\right). \tag{3.37}$$

Note that H is Hermitian, which leads to the same property for G:

$$G^*_{AB} = G_{BA}. \tag{3.38}$$

To implement PMI, the direction of the evolution aligned with the complexity normal $\delta_{\rho_A}C = \frac{\delta C[\rho]}{\delta \rho_A}$ has to be projected out. Therefore, G has to be first modified as

$$G_{AB} \to \tilde{G}_{AB} = G_{AB} - \frac{1}{G_{CD}\delta_{\rho_C}C\delta_{\rho_D}C}\left(G_{AC}\delta_{\rho_C}C\right)\left(G_{DB}\delta_{\rho_D}C\right) \tag{3.39}$$

to the projected "metric". Finally, we need to rewrite the evolution with $\tilde{G}$ and add an additional term for the monotonic increase along the normal direction. This results in

$$\dot{\rho}_A = \tilde{G}_{AB}[\rho_B + \alpha\frac{\delta_{\rho_B}C}{G_{CD}\delta_{\rho_C}C\delta_{\rho_D}C}]. \tag{3.40}$$

α is assumed to be a general positive functional/constant at this point, and it directly

sets the growth of complexity, as from Eq. (3.40)

$$\frac{dC}{dt} = \delta_{\rho_A} C \dot{\rho}_A = \alpha. \tag{3.41}$$

Eq. (3.40) is now our modified evolution equation which keeps the cutoff functional finite, but monotonically growing. Note how PMI has rendered the suggested dynamics Eq. (3.40) *non-linear*. One of the cornerstones of standard QM formulation is widely known to be its linearity, a property which in conjunction with dynamical preservation of Born rule leads to the important property of *unitarity*. In the next section, we will argue that unitarity and/or linearity is not important in the context of the physics of the whole, which this model is claimed to be applicable to. We argue that loss of unitarity in favor of a preferred representation, as with $\rho_{x,y}$ here, is a conceptual advantage, for as explained in Section 2.2.2, unitarity is the source of the basis ambiguity. As long as we have a well-defined statistical theory where probabilities remain normalized dynamically, it is in fact better to dispense with unitarity.

The formulation Eq. (3.36) can potentially obscure the conditions that are to be satisfied. Evolution maps for density matrices must be positive and trace-preserving in at least one representation, both of which are necessary and sufficient for physical interpretation in terms of distribution probability. These two conditions can be verified for the modified equation Eq. (3.40).

Firstly, as for trace-preservation,

$$\operatorname{Tr} \dot{\rho} = \sum_{A_d} \dot{\rho}_A = 0 \tag{3.42}$$

must be satisfied, which A_d denotes diagonal terms. Next, one must show that

$$\operatorname{Tr} \rho P_v \geq 0 \tag{3.43}$$

for a complete set of projectors $P_v := |v\rangle\langle v|$.

Let us assume a representation in which complexity only depends on the diagonal terms, i.e., $C = C[\rho_{A_d}]$. The justification for this assumption comes from the fact that the diagonal terms of density matrix define distribution, which conceptually complexity should be a function of. In the next section, this reasoning will be demonstrated more concretely. Also, note that from Eq. (3.37), $G_{A_d B_d} = 0$ vanishes (setting $a = b, c = d$). So, $G_{A_d B}$ is only non-zero iff B is non-diagonal (in any representation, including the one we are assuming here).

It then follows from Eq. (3.40) that

$$\begin{aligned}\dot{\rho}_{A_d} &= \tilde{G}_{A_dB}[\rho_B + \alpha\frac{\delta_{\rho_B}C}{|\delta C|^2}] \\ &= [G_{A_dB} - \frac{1}{|\delta C|^2}\left(G_{A_dC}\delta_{\rho_C}C\right)\left(G_{DB}\delta_{\rho_D}C\right)][\rho_B + \alpha\frac{\delta_{\rho_B}C}{|\delta C|^2}] \\ &= G_{A_dB}\rho_B - \frac{1}{|\delta C|^2}\left(G_{A_dC}\delta_{\rho_C}C\right)\left(G_{DB}\delta_{\rho_D}C\right)\rho_B \\ &= G_{A_dB}\rho_B,\end{aligned} \tag{3.44}$$

where the other term vanishes for $G_{A_dC}\delta_{\rho_C}C = G_{A_dC_d}\delta_{\rho_{C_d}}C = 0$ due to our assumption about C and the identity demonstrated above. Therefore, it is established that the diagonal elements of the density matrix evolve under Eq. (3.40) in the same manner as in the original model. It then follows that "in this particular representation", where C is dependent on ρ diagonally, both conditions Eqs. (3.42) and (3.43) are *partially* valid (for one particular $\{A_d\}'s$ and set of P_v's), since they only pertain to the diagonal elements.

This result chimes in coherently with the loss of unitarity under implementation of PMI: It essentially breaks the representation equality of QM, and makes the probabilistic interpretation valid only in one particular representation, the same one in which we define ρ_{aa} as the distribution upon which complexity depends. This will be the *position* representation—namely, the representation of fermionic density in the real space. Therefore, a complexity functional of fermionic density in real space picks out a representation in which the modified equation Eq. (3.40) is consistent.

On a different note, this means that implementation of PMI as means to ensure both the generation of relational arrow of time *and* rendering the QFT model naturally regularized, also addresses the problem of preferred basis (see Section 2.2.2) *a fortiori*: It is not limited to merely making a choice on conceptual grounds to fix the representation, but this very choice modifes the dynamics so strongly beyond unitarity that no other representation remains possible. PMI is a strong physical principle, motivated for on conceptual grounds, that addresses the problem of regularization, preferred basis, and arrow of time, all at once, and creates a novel model as in Eq. (3.40) with possibility of novel empirical results. We will discuss this last point with regard to the phenomenology of the model below, once we study proposal(s) for the complexity functional more concretely to show how this general framework is to work.

3.4.2 Explicit complexity functional

Motivated by the N-body system [35], given our interpretation of ρ_{xx} as the spatial distribution of fermions, we can posit the complexity function

$$C[\rho] := \int_{x,y} \frac{\rho_{xx}\rho_{yy}}{|x-y|} \times \sqrt{\int_{x,y} \rho_{xx}\rho_{yy}|x-y|^2}, \tag{3.45}$$

which is a convergent quantity for well-behaved distributions with proper falloff conditions in three dimensions. Eq. (3.45) *is* essentially the N-body complexity function Eq. (2.25) computed for a mass distribution rather than point masses.

We shall now show that Eq. (3.45) satisfies all the key properties we expect from this model once PMI is implemented. Firstly, let us verify that this functional is proportional to high-momentum modes, and therefore, its monotonic growth will effectively regularize the theory by setting a time-dependent cutoff. For simplicity, let us work with a lattice field in a bounded space of size L^D, where D is the dimension. Let a denote the lattice constant. Positions take on the values $x^b = n^b a$, and the boundedness discretizes the fourier modes as $k_n = \frac{\pi n}{L}$ for $n^a = 0, 1, \cdots, N-1$, where $N = L/a$ is the number of modes. Then, assume a wave function composed as superposition of these modes,

$$\psi_x = \sum_n c_n e^{\frac{2\pi i n\cdot x}{L}}, \tag{3.46}$$

where $n \cdot x = n_a x^a = n \cdot m a$ for $x^b = m^b a$, leading to the normalized density matrix

$$\rho_{xx}(x,t) = \frac{1}{N^{2D}} \sum_{n,m} c_m^* c_n e^{\frac{2\pi i(m-n)\cdot x}{L}}. \tag{3.47}$$

Let us only compute the first integral in Eq. (3.45), denoted by $\overline{|x-y|^{-1}}$, which is sensitive to small-scale fluctuations as the result of high modes. It follows

$$\begin{aligned}
\overline{|x-y|^{-1}} &= \frac{1}{N^{4D}} \sum_{x,y} \frac{1}{|x-y|} \sum_{n,m,n',m'} c_{m'}^* c_{n'} c_m^* c_n \, e^{\frac{2\pi i[x\cdot(m-n)+y\cdot(m'-n')]}{L}} \\
&= \frac{1}{2^D N^{4D}} \sum_{u,v} \frac{1}{|u|} \sum_{n,m,n',m'} c_{m'}^* c_{n'} c_m^* c_n \, e^{\frac{2\pi i[u\cdot(\frac{m-n}{2}-\frac{m'-n'}{2})+v\cdot(\frac{m-n}{2}+\frac{m'-n'}{2})]}{L}}, \\
&= \frac{1}{2^D N^{3D}} \sum_{u} \frac{1}{|u|} \sum_{n,m,n',m'} \delta_{\frac{m-n}{2}+\frac{m'-n'}{2},0} c_{m'}^* c_{n'} c_m^* c_n \, e^{\frac{2\pi i[u\cdot(\frac{m-n}{2}-\frac{m'-n'}{2})]}{L}},
\end{aligned} \tag{3.48}$$

where in the second line the new variables $u^a = x^a - y^a, v^a = x^a + y^a$ have been introduced, and in the last line the sum over v has been carried out. Note that $|u| \geq a$. Assuming a maximally superposed state with $c_m = 1$ for all m's yields

$$\begin{aligned}\overline{|x-y|^{-1}} &= \frac{1}{2^D N^{2D}} \sum_u \frac{1}{|u|} \sum_{n,m} e^{\frac{2\pi i[u\cdot(m-n)]}{L}} \\ &> \frac{1}{2^D N^{2D}} \frac{N^D}{a} \sum_{n,m} e^{\frac{2\pi i[a\cdot(m-n)]}{L}} = \frac{1}{2^D a},\end{aligned} \tag{3.49}$$

which is divergent as $a \to 0$. This means that a field configuration with finite C has to be regularized in the high-momentum region *by construction*.

Let us demonstrate the relationship of complexity with cutoff numerically. In a three-dimensional lattice, we randomly generate N modes c_n's for different values of N. This will define a randomly generated density matrix Eq. (3.47). The complexity of this distribution can be computed using Eq. (3.45). Figure 3.1 clearly shows the monotonic increase of complexities with N. Note that $N \propto a^{-1}$, and this shows how complexity itself can act as a scale-invariant short-distance regulator. Moreover, note that the behavior is linear, as shown from our approximate computation in Eq. (3.49).

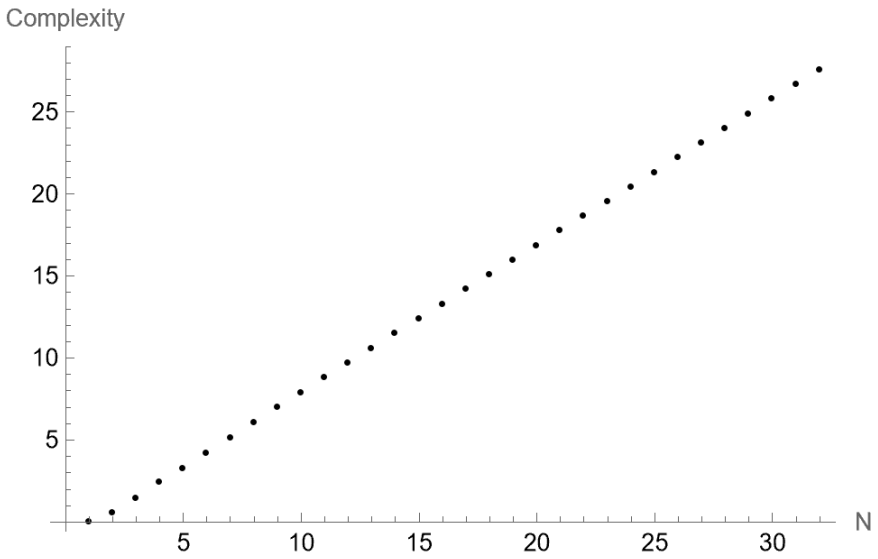


FIGURE 3.1: The complexities of randomly generated density matrices in $3D$ are plotted, from $N = 1$ to $N = 32$. Complexity rises monotonically. Therefore, it is equivalent to the short-distance cutoff (lattice constant).

Secondly, to demonstrate the regularizing aspect more generally, consider the heat-regularization method. Its implementation in kernels is equivalent to regularizing

configurations as

$$\phi \to \phi^\Lambda := e^{\frac{\Delta}{\Lambda^2}}\phi \tag{3.50}$$

to suppress high modes exponentially ($\phi^\Lambda(p) = e^{-\frac{p^2}{\Lambda^2}}\phi(p)$). The complexity of the regularized distribution reads

$$C[\rho^\Lambda] = \int_{x,y} \frac{\rho_x^\Lambda \rho_y^\Lambda}{|x-y|} \times \sqrt{\int_{x,y} \rho_x^\Lambda \rho_y^\Lambda |x-y|^2}, \tag{3.51}$$

which can be derived by inserting Eq. (3.50) in the measure over fields. We need to show that high complexities do in fact correspond to high cutoff values for Λ. In other words,

$$\frac{d}{d\Lambda} C[\rho^\Lambda] > 0 \tag{3.52}$$

must be verified, establishing that C is an equally good cutoff.

Generally, it follows that

$$\begin{aligned}\frac{d}{d\Lambda^2}\int_{x,y} \rho_x^\Lambda A_{x,y} \rho_y^\Lambda &= -\frac{2}{\Lambda^4}\int_{x,y}\left(\Delta\rho_x^\Lambda A_{x,y}\rho_y^\Lambda + \rho_x^\Lambda A_{x,y}\Delta\rho_y^\Lambda\right) \\ &= -\frac{2}{\Lambda^4}\int_{x,y}\left(\rho_x^\Lambda \Delta_x A_{x,y}\rho_y^\Lambda + \rho_x^\Lambda \Delta_y A_{x,y}\rho_y^\Lambda - 2\rho_x^\Lambda \nabla_x\cdot\nabla_y A_{x,y}\rho_y^\Lambda\right),\end{aligned} \tag{3.53}$$

where the last line is by integration by parts. In our case where $A_{x,y} = f(|x-y|) = |x-y|^n$ in both integrals in Eq. (3.45) with $n = -1, 2$,

$$\frac{d}{d\Lambda^2}\int_{x,y} \rho_x^\Lambda A_{x,y}\rho_y^\Lambda = -\frac{8}{\Lambda^4}\int_{x,y}\rho_x^\Lambda \Delta A_{x,y}\rho_y^\Lambda = -\frac{8}{\Lambda^4}\int_{x,y}\rho_x^\Lambda \Delta|x-y|^n\rho_y^\Lambda. \tag{3.54}$$

In three dimensions,

$$\int_{x,y}\rho_x^\Lambda\Delta|x-y|^n\rho_y^\Lambda = \begin{cases} -4\pi\int_x \rho_x^\Lambda\rho_x^\Lambda & n=-1 \\ 6\int_{x,y}\rho_x^\Lambda\rho_y^\Lambda & n=2\end{cases}. \tag{3.55}$$

Finally, this leads to

$$\frac{1}{C[\rho^\Lambda]}\frac{d}{d\Lambda^2}C[\rho^\Lambda] = \frac{8}{\Lambda^4}\left[\frac{4\pi\int_x \rho_x^\Lambda\rho_x^\Lambda}{\int_{x,y}\frac{\rho_x^\Lambda\rho_y^\Lambda}{|x-y|}} - \frac{6\int_{x,y}\rho_x^\Lambda\rho_y^\Lambda}{\int_{x,y}\rho_x^\Lambda\rho_y^\Lambda|x-y|^2}\right]. \tag{3.56}$$

Generally, under the assumption of a large-scale uniform ρ with small-scale fluctuations, the first term to the first degree approximates as

$$\frac{\bar{\rho^2} L^3}{\bar{\rho} L^3 \int_y \rho_y / |y|} \sim \frac{\bar{\rho^2} L^3}{\bar{\rho}^2 L^3 \epsilon^2} \sim \epsilon^{-2}, \tag{3.57}$$

where we have assumed $\bar{\rho^2} \approx \bar{\rho}^2$, as is the case with non-correlated fluctuations. ϵ is the characteristic small length scale within the distribution. As for the second term, it approximates as

$$\frac{\bar{\rho}^2 L^6}{\bar{\rho} L^3 \int_y \rho_y |y|^2} \sim \frac{\bar{\rho}^2 L^6}{\bar{\rho}^2 L^8} \sim L^{-2}. \tag{3.58}$$

Since $\epsilon \ll L$,

$$\frac{1}{C[\rho^\Lambda]} \frac{d}{d\Lambda^2} C[\rho^\Lambda] \sim \frac{1}{\epsilon^2 \Lambda^4} > 0. \tag{3.59}$$

This proves that our complexity functional is *as good as* the heat-kernel regularization, as one well-respected method. This result will be directly used in the next chapter.

Lastly, we would like to show that the proposed complexity functional increases with number of *effective* "particles" within the distribution. Assume Gaussian wave packets $\Psi_{Gx} = \frac{1}{\sqrt{\sigma (2\pi)^{D/2}}} e^{-\frac{x^2}{4\sigma^2}}$. Then for $|\Psi\rangle = \int_{x_1, x_2, \cdots, x_N} \Psi_{Gx_1} \cdots \Psi_{Gx_N} |x_1, \cdots, x_N\rangle$, a generic one-particle density distribution reads

$$\rho_x = \sum_i \frac{c_i}{\sigma_i (2\pi)^{D/2}} e^{-\frac{(x-x_i)^2}{2\sigma_i^2}}, \tag{3.60}$$

where $\sum_i c_i = 1$, and represents a distribution of point-particles at locations x_i's. In unbounded three-dimensional space, we generally can evaluate the two-particle averages

$$\int_{x,y} \frac{c_i c_j}{\sigma_i \sigma_j (2\pi)^3} e^{-\frac{(x-x_i)^2}{2\sigma_i^2} - \frac{(y-x_j)^2}{2\sigma_j^2}} |x-y|^n \sim c_i c_j |x_i - x_j|^n + O(\sigma). \tag{3.61}$$

Therefore,

$$C[\rho] \sim \sum_{i \neq j} \frac{c_i c_j}{|x_i - x_j|} \sqrt{\sum_{i \neq j} c_i c_j |x_i - x_j|^2}, \tag{3.62}$$

which is the N-body complexity function for point-particles with masses c_i's. To find

a lower bound, let us assume a minimal-complexity distribution of N particles in a uniform spherical distribution. The largest contribution to the first term comes from short distances ϵ between particles, which are assumed to be approximately the same in this uniform distribution. The second term measures the large distances, for which we can consider all the separations between the anti-podal particles on the outermost shell, which are all same as diameter of the sphere, $2L$. Generally, $N^{\frac{2}{3}}$ are expected to fill the outermost shell, and there are $\frac{N(N-1)}{2} \sim \frac{N^2}{2}$ inter-particle separations. Thus, it follows

$$C[\rho] > \frac{N^2}{2\epsilon}\sqrt{2N^{\frac{2}{3}}L^2} \tag{3.63}$$

for large N. Given the radius of the spherical distribution is $L/\epsilon \sim N^{\frac{1}{3}}$, we conclude

$$C[\rho] > AN^{\frac{8}{3}}. \tag{3.64}$$

Therefore, C rises as the distribution effectively defines more and more localized structures in the distribution counting as "particles".

The last property shows that in a fundamental QFT model of the universe, the rise of complexity, formation of localized subsystems, and the intrinsic cutoff of the theory are all unified. In particular, it means that creation of particles in the *whole* universe is regulated by the same intrinsic cutoff. This is a novel holistic aspect to our model, as can be seen directly from the non-locality of Eq. (3.40). However, note that in the particular fermionic model studied here, the total number is conserved which renders this result irrelevant. This is not true in general. Although charge is conserved in QED, number of fermions is not, and this highlights the significance of what we have established.

We argue that the candidate Eq. (3.45) is ideal as a first choice, for it not only satisfies all conceptual criteria, it keeps the QFT model in unity with the classical model and thereby provides a smooth classical limit. But we can and must remain open to other choices, if we are to make novel applicable physics. Different choices for complexity functionals affect only the high energy description of the QFT model, for as long as $C \gg 1$, which is the case in the later-Universe epoch where we are studying effective physics, we already have lots of modes turned on. Irrespective of the physics close to the cutoff (where different C's describe differently), the low-energy ($p \ll \Lambda$ in standard QFT) physics remains the same, as standard renormalization shows [32].

3.5 Phenomenology of the model

We ought to investigate what empirical aspects to the history of the universe are captured by the fermionic model presented here. The one-particle reduced density matrix Eq. (3.47) is capable of modeling the distribution of structures in the universe, which are all fermionic. However, by construction, the correlations within entangled states are lost and "flattened out" in the one-particle sector of the Fock space. This poses some limitations to our empirical interpretation, and requires a complementary principle for measurement. To see what the problem is, consider a simple two-particle entangled fermionic state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|1,1'\rangle_A + |2,2'\rangle_A), \tag{3.65}$$

where A denotes anti-symmetrization. The density matrix reads

$$\begin{aligned}\rho = \frac{1}{2}\,&(|1,1'\rangle_A \langle 1,1'|_A + |2,2'\rangle_A \langle 2,2'|_A \\ &+ |1,1'\rangle_A \langle 2,2'|_A + |2,2'\rangle_A \langle 1,1'|_A)\,.\end{aligned} \tag{3.66}$$

The one-particle reduced density matrix Eq. (3.47) follows from projection onto one-particle space:

$$\rho_1 = \frac{1}{4}\,(|1\rangle\langle 1| + |2\rangle\langle 2| + |1'\rangle\langle 1'| + |2'\rangle\langle 2'|)\,, \tag{3.67}$$

which represents a flattened-out distribution of all posssible states that one might see in a two-particle system, regardless of entanglement correlations.

The result established in this simple two-particle system carried over into a general model of measurement. As explained in Section 2.2.2, under joint evolution of the system and environment containing measuring apparatus, the system becomes entangled as

$$|\psi_s\rangle\,|E\rangle \to \sum_i |\psi_s\rangle_i\,|S\rangle_i\,, \tag{3.68}$$

with determinate outcomes registered in each term. Therefore, the entangled distribution is central to generation of individual outcomes. Flattening the correlations within the entangled state of the system with apparatus destroys individual outcomes.

Notwithsanding the loss of entanglement, we claim to have a quantum field model of gravitational formation of structure in the universe, as the model primarily originates

from exploring the non-relativistic weak limit of a quantum gravity model. Also, the increase of complexity as a measure of structure formation is implemented. Therefore, there is no question that localized structures dynamically form within an originally homogeneous distribution. Once they do, we can define reference units of length and time.

Moreover, given the inherent position representation, there is no ambiguity as to how to interpret structures. Within these localized structures, macroscopic subsystems with large degrees of freedom will emerge along with subsystems with much fewer ones. The former can be interpreted as measuring apparatus, and the latter as "quantum systems". However, this model is not capable of producing the entanglement between apparatuses and system, as demonstrated above, and therefore, it it limited as to give a theory of measurement.

In spite of settling the preferred-basis problem, this model does not address the measurement problem. Once apparatus and system have been identified and during a measurement process are entangled, we need to invoke an external agent that effectively "selects" a particular branch of the full decohered wave function, same as the standard QM. This selection needs to be carried out prior to reduction to the one particle density matrix ρ_{ab}; otherwise, the result of the measurement (entanglement between the apparatus and system) will be lost. As a result, this selection effectively collapses the wave function. Afterward, the density matrix describes the realized distribution of fermionic degrees of freedom that comprise the formation of the record of the measurement.

Chapter 4

Bosonic field theory: A de Broglie–Bohm Model

"In 1952 I saw the impossible done. It was in papers by David Bohm."

John Bell [43]

4.1 Motivation and setup

If we are to deliver a complete framework, we have to extend the principles of our work to Bosonic field theories, which play a necessary role in interacting theories. As proposed in the previous chapter, on solid ground it is natural to posit fermionic charge density as a minimal fundamental ontology from which all structures of empirical phenomena can be built up. The coupling with Bosonic fields need not obscure this. Our strategy in constructing an interacting theory is to use Bosonic fields not as independent degrees of freedom, but as a *bookkeeping* tool which encodes the actual charge distribution, available for extraction at each instant.

Next, in order to have a conceptually well-defined quantum model, as well as acquire an actual realized configuration, we invoke de Broglie–Bohm model. de Broglie's work on an alternative quantum theory in 1927 [37], which was incorporated into Bohm's independent rediscovery of the same tenets within standard QM [17], still stands out to this date. It is the simplest model which describes quantum phenomena as microscopic processes objectively unfolding in configuration space. Therefore, it addresses both the preferred basis problem and the measurement problem at once. de Broglie–Bohm

(dBB) model is briefly discussed in Appendix A.

dBB model can be constructed for any canonical quantum model, including field theory. We shall study its application to the bosonic field theory in which Dirac fermions interact via Abelian $U(1)$ gauge theory. This is sufficient to describe much of the atomic and molecular structure formation in the full relativistic limit. In addition to its heuristic role in exemplifying the implementation of our principles and completing PSDQFT, the proposed dBB bosonic model does definitely convey features of our actual Universe. In particular, it describes non-gravitational structure formation driven primarily by electromagnetism. Thus, we can think of it as describing a universe in which large-scale gravitational subsystems have already formed, and electromagnetism takes over at smaller scales to drive further formation.

Consider the classical electromagnetic action coupled to Dirac fields:

$$S[\psi,\bar{\psi},A_a,A_0] = \int dt \int_x \left(-\frac{1}{2}F_{0a}F^{0a} - \frac{1}{4}F_{ab}F^{ab} + \psi^\dagger(i\partial_t - D)\psi + e\psi^\dagger\gamma^a\psi A_a + e\psi^\dagger\psi A_0\right), \tag{4.1}$$

where $F_{0a} = \dot{A}_a - \partial_a A_0$, and $F_{ab} = \partial_a A_b - \partial_b A_a$, and A_0, A_a are the components of the gauge fields, and we are deliberately separating space from time. Finally, D is the Dirac operator (Eq. (3.18)). To carry out the Legendre transformation, let us denote the conjugate momenta to $A_a, A_0, \psi^\alpha, \psi^\dagger_\beta$ by $\Pi^a, \Pi^0, p_{\psi^\alpha}, p_{\psi^\dagger_\beta}$ respectively. It follows that

$$\begin{aligned}
\Pi^a(x) &= \frac{\delta S}{\delta \dot{A}_a(x)} = \dot{A}^a,\\
\Pi^0(x) &= \frac{\delta S}{\delta \dot{A}_0(x)} = 0,\\
p_{\psi^\alpha}(x) &= \frac{\delta S}{\delta \dot{\psi}^\alpha(x)} = i\psi^\dagger_\alpha,\\
p_{\psi^\dagger_\beta}(x) &= \frac{\delta S}{\delta \dot{\psi}^\dagger_\beta(x)} = 0.
\end{aligned} \tag{4.2}$$

The Hamiltonian reads

$$\begin{aligned}
H &= \int_x \left(\Pi^a\dot{A}_a - \mathcal{L}\right)\\
&= \int_x \left(\frac{1}{2}\Pi^a\Pi_a + \frac{1}{4}F^{ab}F_{ab} - A_0(\partial_a\Pi^a - \rho) - A_a j^a + H_D\right).
\end{aligned} \tag{4.3}$$

ρ and j^a are the charge and current densities, which for the Dirac field are

$$\rho = e\psi^\dagger\psi, \qquad j^a = e\psi^\dagger\gamma^a\psi. \tag{4.4}$$

And H_D denotes the Dirac Hamiltonian:

$$H_D = \psi^\dagger D\psi. \tag{4.5}$$

The Poisson structure (with anti-commutation for Dirac fields) reads

$$\begin{aligned} \{A_a(x), \Pi^b(y)\} &= \delta_a^b \delta^3(x-y), \\ \{\psi^\alpha(x), \psi^\dagger_\beta(y)\}_+ &= -i\delta^\alpha_\beta \delta^3(x-y). \end{aligned} \tag{4.6}$$

Finally, the theory is constrained, with two first-class constraints

$$\begin{aligned} \Pi^0 &\approx 0, \\ \partial_a \Pi^a - \rho &\approx 0, \end{aligned} \tag{4.7}$$

linear in momenta, that generate gauge transformations for A_0 and the longitudinal component of A_a. The second one is the Gauss constraint. Note that we have implicitly applied the constraints corresponding to momenta conjugate to Dirac fields. Therefore, the Dirac sector of the model is already in its reduced phase space.

We should now study the dBB quantization of the model aligned with the conceptual pictured laid out above.

4.2 Quantization and de Broglie–Bohm model

The dBB quantization of gauge systems is developed in Appendix A in two approaches. The first is the Dirac approach, in which first-class constraints are quantized and imposed as time-independent operator equations on the wave function. This method is the most compatible with the dBB model, provided that the constraints are linear in the momenta, so that the gauge orbits close on configuration space and remain momentum-independent. The second approach involves using the reduced phase space of the classical theory and then quantizing only the physical degrees of freedom. In this case, the associated dBB configurations are entirely physical (gauge-invariant).

Here, given we are interested in the charge density encoded in the longitudinal component of A_a via Gauss constraint, the reduced phase space approach is of no significance. If we simply gauge away the longitudinal mode and develop a well-defined dBB dynamical model of transverse configurations, we will lose the connection through this constraint, by construction. Therefore, let us implement Dirac approach, and quantize the theory Eqs. (4.3) and (4.7).

We posit the commutator and anti-commutator

$$\begin{aligned}[\hat{A}_a(x), \hat{\Pi}^b(y)] &= i\delta_a^b \delta^3(x-y) \\ [\hat{\psi}^\alpha(x), \hat{\psi}^\dagger_\beta(y)]_+ &= \delta^\alpha_\beta \delta^3(x-y),\end{aligned} \tag{4.8}$$

with all the other commutator and anti-commutators vanishing. In configuration space basis, we can represent them as

$$\begin{aligned}\hat{\Pi}^a(x) &\equiv -i\frac{\delta}{\delta A_a(x)}, \\ \psi^\dagger_\alpha(x) &\equiv \frac{\delta}{\delta \psi_\alpha(x)}.\end{aligned} \tag{4.9}$$

Quantizing the Hamiltonian in Eq. (4.3) using these relations lead to Schrödinger equation

$$i\partial_t \Psi = \hat{H}\Psi. \tag{4.10}$$

Also, Dirac approach requires the wave function to satisfy the constraints:

$$\begin{aligned}-i\frac{\delta}{\delta A_0(x)}\Psi &= 0, \\ -i\partial_a \frac{\delta}{\delta A_a(x)}\Psi &= \hat{\rho}(x)\Psi.\end{aligned} \tag{4.11}$$

The first relation declares A_0-independence of Ψ, while the second one constrains the dependence of the wave function on longitudinal modes.

The transition to a de Broglie–Bohm model is straightforward. We posit the guidance equation for the "actualized" configuration $\bar{A}_a$, given the wave function $\Psi = Re^{iS}$. Note

$$\frac{\delta}{\delta A^a} S = \frac{\delta}{\delta A^a} \mathrm{Im} \log \Psi = \frac{\mathrm{Im}\, \Psi^* \frac{\delta}{\delta A^a}\Psi}{\Psi^*\Psi}. \tag{4.12}$$

The bookkeeping role of $\bar{A}_a$ is key here. Therefore, we demand that the only realized configurations be electromagnetic fields to read off the charge density from it. Therefore, we do not implement the full dBB equations, and will write down a "partial" guidance equation for bosonic fields only, averaging out the fermionic configurations w.r.t the Born measure. That is,

$$\dot{\bar{A}}_a = \int_F \frac{\delta S}{\delta A^a}\bigg|_{A_a=\bar{A}_a} \times \frac{\Psi^*\Psi}{\int_F \Psi^*\Psi}\bigg|_{A_a=\bar{A}_a} = \frac{\int_F \mathrm{Im}\, \Psi^* \frac{\delta}{\delta A^a}\Psi}{\int_F \Psi^*\Psi}\bigg|_{A_a=\bar{A}_a}, \tag{4.13}$$

where $\int_F \equiv \int D[\psi^\dagger]D[\psi]$ denotes integration over all fermionic degrees of freedom, and we have essentially taken an average over fermionic fields according to Born measure $\Psi^*\Psi$, rendering the whole expression for $\dot{\bar{A}}_a$ independent of fermionic fields.

Finally, let us establish the said bookkeeping relation. One must simply take the divergence of the guidance velocity field in Eq. (4.13) and make use of the second equation in Eq. (4.11), resulting in

$$\partial_a \dot{\bar{A}}^a(x) = \left.\frac{\int_F \mathrm{Im}\,\Psi^* \partial_a \frac{\delta}{\delta \bar{A}_a(x)}\Psi}{\int_F \Psi^*\Psi}\right|_{A_a=\bar{A}_a} = \left.\frac{\int_F \mathrm{Im}\,\Psi^* i\hat{\rho}(x)\Psi}{\int_F \Psi^*\Psi}\right|_{A_a=\bar{A}_a} = \left.\frac{\int_F \Psi^*\hat{\rho}(x)\Psi}{\int_F \Psi^*\Psi}\right|_{A_a=\bar{A}_a} \equiv \langle\hat{\rho}(x)\rangle_{\bar{A}_a}, \tag{4.14}$$

where the Hermiticity of $\hat{\rho}$ is used. $\langle\hat{\rho}\rangle_{\bar{A}_a}$ means the expectation value of charge density *conditioned upon* the realized configuration. This delivers the stated purpose of the partial dBB model and nature of the bosonic fields. Eq. (4.13), assuming the RHS is integrable (see the next section), gives a dynamical dBB electromagnetic field configuration. At each instant, the divergence gives us the fermionic charge density consistent with the conditional wave function.

4.3 Complexity regularization

Same with any other continuum QFT propagator kernel, the evolution governed by Eq. (4.10) needs to be regularized, for as it stands, it is ill-defined. In the canonical picture, complementary to the path-integral formulation, the issue boils down to the Born measure $|\Psi[\phi]|^2$ and in effect, the self-adjointness of the Hamiltonian (see Section 2.2.4). The relation $\langle\Psi'|\hat{H}|\Psi\rangle$ has to be well-defined. In momentum representation, as long as there is a cutoff and the inner product effectively sees only a finite domain, i.e.,

$$\langle\Phi'|\Phi\rangle = \prod_{p\leq\Lambda} \delta(\Phi'_p - \Phi_p), \tag{4.15}$$

$\hat{H}$ remains self-adjoint, and the propagator $U = e^{-itH}$ is unitary and norm-preserving. But in the continuum limit, field configurations can vary significantly over infinitesimally small regions, which spoils the absolute continuity of the Born measure. This means that even if we consider initially well-behaved normalizable wave functions with support over field configurations with finite fourier modes, under evolution, the corresponding Born measure will develop infinities. This renders the Hamiltonian-evolution non-unitary, and the Hamiltonian itself, non self-adjoint.

Beyond this "standard" knowledge, it is vital to emphasize that on first principles, it seems we can circumvent this issue in the dBB approach. The reason is that one is primarily interested in solving the guidance equation Eq. (4.13). As long as it remains integrable, we can evolve the uniquely realized configuration, irrespective of whether or not the Born measure associated with the whole wave function is well-defined (unless one is also interested in equivariance and positing the Born measure as the probability distribution for the realized configurations).[1] In other words, the notorious infinities under time evolution may simply cancel out in the guidance equation.

Therefore, it is a valid question at this stage that whether dBB QFT is even in need of regularization, so long as one is primarily interested in integrating the guidance equation. Note that regardless of the answer, we do need at least a regularized theory w.r.t Dirac fields (e.g., either by regularizing Dirac fields or regularizing Dirac kinetic term). It is because we are implementing partial guidance equation, and require a well-defined Born statistics over Dirac fields, as it appears in Eq. (4.13).

Regardless, it can be shown that the very integrability requirement is equivalent to the usual regularizability.[2] This means that a well-defined guidance equation imposes the same conditions for the quantum fields as the be regularized. Therefore, even in the dBB approach, we need to regularize the model.

As shown in [45], the problem boils down to the representation of the field momentum Eq. (4.9). Given the kinetic term in the action is

$$T = \int_x \sum_a \frac{1}{2}(\dot{A}_a - \partial_a A_0)^2 \equiv \int_{x,y} \sum_a \frac{1}{2}\dot{A}_a(x) g_{xy} \dot{A}_a(y), \tag{4.16}$$

there is an implicit kinematic metric which we glossed over, that is $g_{xy} = \delta^3(x-y)$. The problem originates from this point-coincident metric, and one needs to regularize it to have well-defined guidance equation. This is the standard regularization of quantum fields. As with the fermionic density model, we would like to implement regularization w.r.t the same complexity function explored in Section 3.4.2.

Assume a well-defined (regularized) kinetic metric ${g^\Lambda}_{xy}$, with cutoff Λ (to be specified later) that leads to a regularized QFT with finite Born measure. We assume that in the limit $\Lambda \to \infty$, the metric goes to the point-coincident $\delta^3(x-y)$. The

[1] The question of how Born statistics comes out of dBB model is subtle, and can be answered in more than one way beyond the standard approach that argues from equivariance. See [44]. We do not enter this topic and assume Born statistics has already been established through one of the possible mechanisms.

[2] The proof was communicated in private in an unpublished note by Tim Koslowski.

momentum conjugate to the fields reads

$$\Pi^a(x) = \int_y g^{\Lambda}{}_{xy}\dot{A}^b(x), \tag{4.17}$$

and the kinetic term in the Hamiltonian becomes

$$T = \int_{x,y} \Pi_a(x) g^{\Lambda}{}^{-1}_{xy} \Pi_a(y), \tag{4.18}$$

where the inverse of the metric has appeared. This means that the guidance equation Eq. (4.13), where we implicitly used Dirac delta function, needs to change accordingly to:

$$\dot{\bar{A}}_a(x) = \left. \frac{\int_y \int_F g^{\Lambda}{}^{-1}_{xy} \mathrm{Im}\, \Psi^* \frac{\delta}{\delta A^a(y)} \Psi}{\int_F \Psi^* \Psi} \right|_{A_a = \bar{A}_a}. \tag{4.19}$$

Also, given Eq. (4.17), the expression Eq. (4.14) changes to

$$\partial_a \int_y g^{\Lambda}{}_{xy} \dot{\bar{A}}^a(y) = \langle \hat{\rho} \rangle_{\bar{A}_a}(x). \tag{4.20}$$

Everything else remains the same.

Hence, Eq. (4.19) is a well-regularized guidance equation. As for the choice of the metric, one standard option is the heat-kernel regularization Eq. (3.50). However, given our conceptual insistence on using a universal complexity functional as regulator, and also in light of our analysis of the equivalence of this method with the complexity-regularization in Section.3.4.2, we use complexity as the cutoff:

$$g^{C}{}_{xy} = \langle x | e^{\frac{\Delta}{C}} | y \rangle. \tag{4.21}$$

We simply use the charge density in the definition, Eq. (3.45),

$$C = C[\partial_a \dot{\bar{A}}^a] \equiv C[\langle \hat{\rho} \rangle]. \tag{4.22}$$

This makes Eq. (4.19) a non-local (LHS at x is coupled to different y's in the RHS) dynamical equation for the gauge fields, regularized by the value of the complexity functional of the charge density that it defines at each point along the evolution.

A potential concern might be the behavior of complexity —Here, our natural cutoff —along the dynamics. If complexity itself becomes divergent at a finite "physical"

time[3], it will ruin the physical significance of the cutoff. From Eqs. (4.20) and (4.22), it follows

$$\begin{aligned}\frac{dC}{dt} = \int_x \frac{\delta C}{\delta\langle\hat{\rho}\rangle(x)}\frac{d\langle\hat{\rho}\rangle(x)}{dt} &= \int_{x,y} \frac{\delta C}{\delta\langle\hat{\rho}\rangle(x)}\partial_{x^a} g^C{}_{xy}\ddot{\bar{A}}^a(y) \\ &= \int_x -\frac{\delta C}{\delta\langle\hat{\rho}\rangle(x)}\partial_a \int_F \frac{\delta V_T}{\delta A_a(x)}\bigg|_{A_a=\bar{A}_a},\end{aligned} \tag{4.23}$$

where $V_T = V + V_Q$ is the total potential including the quantum potential V_Q (see Section A.1 in Appendix. A). dC/dt remains generically finite even w.r.t the nominal Newtonian time t.

Although we are now in possession of a regularized bosonic QFT, if we are to fulfill the whole "time as a regulator" idea perfectly, we ought to implement PMI as well. It is reasonable to assume complexity has a *secular* growth in this model.[4] Nevertheless, PMI demands its *monotonic* increase. This will not only identify complexity with time, but address the above concern *a fortiori*.

[3]In keeping with our relational principles, of course, we should be referring to time measured by physical clocks within the universe described by the fermionic charge density. This is something distinct from the nominal Newtonian time we have been using in construction of this model. However, for the sake of the argument and in light of what follows in the analysis, we assume t marches in step with any such physical time. This means that there is a linear relation between them.

[4]We state this conjecture without proof. At least for the coherent states where the configuration follows the classical trajectory, electromagnetism behaves largely like gravity in the non-relativistic limit, and we can assert the secular growth.

Chapter 5

Conclusion

This work comprises two major themes, each treated separately yet developed in unity. The first concerns the foundational analysis of what is at stake in Quantum Theories in general, and in Quantum Field Theory in particular. In Section 2.2, we distilled the key conceptual challenges into four distinct problems and delineated each in turn. The first two are general quantum-theoretical issues, independent of any particular ontology or model construction. The first, and perhaps most (in)famous, is the measurement problem, or, as refined in our discussion, the reality problem. The second, closely related, is the representation problem. To summarize concisely, the measurement problem concerns the failure of QM to function as a universal theory of nature encompassing subsystems, measuring devices, and observers alike. The representation problem, in contrast, pertains to how QM enforces a symmetry in the formalism—namely, unitarity—which is not based on empirical observation. The question is what precisely breaks this symmetry in the emergence of definite outcomes, and how.

The third problem is interpretational: it arises from the difficulty of ascribing a clear physical meaning to the wave function ψ. We discussed how this difficulty intensifies in QFT, where both particle and field notions are intertwined. Finally, we identified and analyzed the most technical problem of QFT—namely, ultraviolet (UV) divergences—and traced their origin.

Following this critical assessment, we introduced the principles of Pure Shape Dynamics in Section 2.3, which constitute the conceptual foundation of this work. In Section 2.4, we anticipated that the PSD framework could resolve most, if not all, of the aforementioned issues. The overarching idea is that shapes define a preferred ontology, thereby resolving both representation and interpretation problems.

Moreover, identifying relational time with the complexity of shapes introduces an intrinsic cutoff, rendering the QFT model regularized from the outset. Crucially, the resolution of the representation and interpretation problems is not merely a matter of "preference" for shapes of fermionic distributions. As demonstrated in Section 3.4.1, the modified fermionic model, conceived of as a statistical model of density function, breaks the unitarity, and is consistent exclusively with the position representation, in which the shape distribution is defined. Thus, the formalism itself uniquely picks out the representation.

The second major theme of this work, being more technical, concerns the construction of two explicit models designed to illustrate how the conceptual framework of PSD manifests in concrete physical systems. We developed both a fermionic and a bosonic field model. Each arises as a particular limit of a general interacting theory in which the fermionic distribution plays a fundamental role in the physical description. The first model, presented in Chapter 3, was motivated based on the non-relativistic, weak-field regime of a general putative quantum theory of gravity. The second model, developed in Chapter 4, involves electromagnetic interactions within a charged fermionic density. The latter implements the dBB formulation, whose principal advantage lies in its natural resolution of the measurement problem.

Despite their differences, both models share the same architectonic features characteristic of the PSD framework. Specifically, it was first demonstrated how one may begin from a general QFT model of fields and derive an equation for the fermionic density (matrix), $\langle\hat{\psi}^\dagger\hat{\psi}\rangle$, in the first model, and $\langle\hat{\psi}^\dagger\hat{\psi}\rangle_{A_a=\bar{A}_a}$ in the second. In both cases, the *necessity* of regularization was established, and each model was regularized with respect to the same complexity functional of the fermionic density introduced in Eq. (3.45).

To verify the consistency of our proposed mechanism for resolving UV divergences, we examined in Section 3.4.2 the relation between the complexity functional and the small-scale cutoff in QFT. Complexity regularization is viable only if finite complexity entails a finite cutoff. We verified this connection using both lattice and heat-kernel regularization schemes. In the former, fields are discretized according to a lattice constant, and it was shown, analytically and computationally, that complexity increases monotonically with the inverse lattice constant. In the latter, fields remain continuous while high Fourier modes are exponentially suppressed; again, complexity was shown to increase monotonically with the heat-kernel cutoff.

In the fermionic model, we further modified the governing equation to implement *Principle II* of PSD. The resulting equation, Eq. (3.40), enforces a monotonic increase of complexity throughout the evolution, thereby allowing progressively higher modes

to become dynamically active.

This initial investigation opens several avenues for further development. First, the modified fermionic model should be studied in greater depth to determine its empirical deviations from the standard formulation Eq. (3.32) and to identify the novel physical effects it predicts. Second, the implementation of PMI within the dBB model of Chapter 4 remains to be completed. We propose employing the same strategy of decomposing the dynamics into two components and modifying the tangential to complexity to ensure a monotonic increase.

Of even greater significance is extending this framework toward a Quantum Field Theory of gravity, that is, Quantum Gravity. The fermionic model studied here was motivated by the conformal sector of gravity, but a more comprehensive model is required to account for the remaining transverse degrees of freedom. These are encoded in the two local degrees of freedom of 3D conformal geometry, represented as the scale-invariant part of the metric, $\rho_{ab} = g_{ab}/|g|^{\frac{1}{3}}$. Current efforts to quantize the classical PSD model of conformal geometry [6] are ongoing, and several ideas developed in this work are expected to be relevant. In particular, we think that complexity is likely to play a central role in introducing a natural cutoff for higher modes of gravity. This mechanism would provide a complexity-based cutoff for the transverse gravitational modes in the classical theory, as well as for quantum fields in the corresponding quantum model. These directions remain under active investigation.

Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten und nicht veröffentlichten Schriften entnom- men wurden, sind als solche kenntlich gemacht. Die Arbeit ist in gleicher oder ähnlicher Form oder auszugsweise im Rahmen einer anderen Prüfung noch nicht vorgelegt worden. Ich versichere, dass die eingereichte elektronische Fassung der eingereichten Druckfassung vollständig entspricht.

Köln, den 27. Oktober 2025

Pooya Farokhi

Appendix A

Constrained Systems and dBB Quantization

In 1952, David Bohm proposed a realistic model of QM (in the sense of describing quantum phenomena as objective physical processes) [17]. This model builds upon de Broglie's alternative quantum theory in 1927 [37], in which particles are guided by a Schrödinger-type wave. Under suitable assumptions, Bohm's model empirically reproduces the Born-rule statistical predictions of the standard interpretation. We shall briefly review this model, particularly from the perspective of the Bohmian interpretation. This will be followed by a section on the application of the model to constrained systems.

A.1 Review of de Broglie–Bohm model

We shall start from the Schrödinger equation. Assume the time-dependent equation is given for the function $\psi(q,t)$ defined on the configuration space $\mathcal{Q}$, equipped with a metric g_{ab}:

$$i\partial_t\psi = \hat{H}\psi = -\frac{1}{2m}\Delta\psi + V(q)\psi, \tag{A.1}$$

where $\Delta = g^{ab}\nabla_a\nabla_b$ is the Laplace-Beltrami operator, and $V(q)$ is a scalar potential on $\mathcal{Q}$.

For the following construction, we perform a polar decomposition of ψ as $\psi = Re^{iS}$, where R and S are real functions and $R(q,t)$ is constrained to be strictly positive. What defines the de Broglie–Bohm (dBB) model, at least in the interpretation we adopt, is the postulate that:

1. For any solution to Eq. (A.1), there exists a unique time-parametrized trajectory $\gamma : \mathbb{R} \to Q$, interpreted as the actual micro-evolution of a quantum system in its configuration space.

2. This "dBB realized trajectory" obeys the dynamical equation

$$\dot{\gamma}^a(t) = \frac{1}{m}\,\partial_a S|_{q=\gamma(t)\equiv Q} = \frac{1}{m}\,\frac{\mathrm{Im}\,\psi^*\partial_a\psi}{\psi^*\psi}\bigg|_{q=\gamma(t)\equiv Q}, \tag{A.2}$$

   referred to as the "guidance equation". The system Eqs. (A.1) and (A.2) completes the construction of the dBB quantum model. Once the initial conditions are set, together they determine the unique evolution of a dBB configuration, Q.

As Bohm demonstrated, this system of equations can be interpreted as a classical system with quantum modifications. The equation for S derived from Eq. (A.1) reads

$$\partial_t S + \frac{1}{2m}(\nabla S)^2 + V(q) - \frac{\Delta R}{2mR} = 0, \tag{A.3}$$

which is the Hamilton–Jacobi partial differential equation with an additional term, the so-called "quantum potential":

$$V_Q := -\frac{\Delta R}{2mR}. \tag{A.4}$$

Thus, the dBB dynamical system defines trajectories that follow the classical dynamics modified by the R-dependent quantum potential.

Regarding statistical predictions, one must explain the origin of the Born rule. There are different approaches to this problem [44]. For brevity and relevance, we present one such strategy below from [46].

We posit that the initial condition of the entire semi-autonomous, effectively isolated subsystem (including the apparatus and quantum systems) is chosen to be typical with respect to the initial Born measure $R^2(q)|_{t_{\text{initial}}}$. Consequently, the joint dynamical evolution Eqs. (A.1) and (A.2) preserves the Born statistics under the guidance equation. From the Schrödinger equation, R evolves as

$$\partial_t(R^2) + \frac{1}{m}\nabla\cdot(R^2\nabla S) = 0, \tag{A.5}$$

which is simply the conservation law for R^2 under the velocity field $\frac{1}{m}\nabla S$, i.e., the guidance velocity. This property is called "equivariance", meaning that the Born measure dynamically evolves in the same manner as any probability measure on Q,

conserved under guidance dynamics.

The kinematical postulate of initial typicality, together with the dynamical property of equivariance, ensures that the statistics of realized dBB subsystems satisfy Born statistics [46]. Note that, strictly speaking, the dynamical system Eq. (A.2) by itself does not require the construction of a Hilbert space with an inner product. It only requires the integrability of equations along a trajectory. However, if Born statistics are to be reproduced, one must assume that R^2 remains normalizable. In the case of fields, this distinction becomes more significant: normalizability of the wave functional is equivalent to field regularization.

The dBB model also provides a solution to the two general conceptual problems of QM mentioned in Section 2.2: the reality problem and the representation problem. Regarding the former, there is no collapse postulate in the system. The dBB model is defined solely by the Schrödinger equation and the guidance equation, and an actual configuration exists with well-defined values for all observables at each instant, regardless of any act of observation. Collapse is effectively replaced by the fact that the realized configuration selects a branch. For example, assume the total wave function of the system and apparatus in the environment, ψ_T, is given. Under decoherence [22], the system effectively evolves into

$$\psi_T \to \sum_i \psi_{s,i}, \psi_{\text{env.},i}, \tag{A.6}$$

where the $\psi_{\text{env.},i}$'s have mutually disjoint support. Even after decoherence, the measurement problem reduces to which branch is realized during observation. In the dBB model, no wave function collapse is required: The realized configuration Q_T occupies one of the branches, registering the corresponding pointer state as the outcome, while all other branches remain empty and irrelevant. The existence of this micro-dynamics resolves the measurement problem.

Regarding the representation problem, the dBB model fixes the representation from the outset by defining a dynamical system on configuration space. It is not merely a matter of preference: The dBB model, capable of reproducing Born statistics, is incompatible with alternative representations, such as momentum space or hybrid configuration-momentum spaces [47]. This restriction arises because the kinetic term is quadratic, whereas higher-degree potentials can spoil the consistency of the dynamical system and conservation of Born statistics.

A.2 The application to constrained systems

dBB quantization, characterized by the joint system in Eqs. (A.1) and (A.2), can be constructed for any constrained system, provided that one of the standard quantization schemes applies.

By a constrained system, we mean a canonical Hamiltonian $H = p_a\dot{q}^a - L$ defined on the phase space $\mathcal{P} = T^*\mathcal{Q}$, subject to a set of irreducible constraints $C_a = 0$, such that $\{C_a, C_b\}$ is of maximal rank. Furthermore, we assume that the system is dynamically consistent, meaning that $\dot{C} = \{C, H\}$ vanishes on the specified constraint surface. Following Dirac's analysis of constrained systems [13], we classify them into two categories. The first category, called "first-class" constraints, are defined as those whose Poisson brackets vanish on the constraint surface, i.e.,

$$\{\gamma_a, \gamma_b\}|_{C_a=0} = 0. \tag{A.7}$$

The second category of constraints, called "second-class", consists of an even number of constraints whose Poisson structure $\{\chi_a, \chi_b\}|_{C_a=0}$ has maximal rank.

As Dirac showed, the total Hamiltonian that generates the dynamics can be written as

$$H_T = H + v^a\gamma_a, \tag{A.8}$$

where v's are arbitrary coefficients. It is then evident that first-class constraints generate gauge-transformations along motion, corresponding to arbitrary variation in v's.

There are two general approaches to the quantization of constrained systems, excluding other methods such as BRST quantization and the Fock–Dirac approach. The first is reduced phase space quantization, which quantizes the classical system on its reduced phase space, where all constraints are satisfied and the Dirac bracket serves as the projected symplectic structure. Regardless of the dBB construction, reduced phase space quantization is typically plagued by mathematical difficulties and ambiguities, as discussed below.

The second approach is Dirac's method, which applies to systems with first-class constraints[1] that generate gauge transformations. We briefly introduce each approach and examine their application to the dBB model.

[1]This does not limit the generality of the method, since all second-class constraints can be formulated as gauge fixations of a larger system containing only first-class constraints.

A.3 Reduced phase space quantization

In the reduced phase space method, the constraints are imposed at the classical level, and the resulting unconstrained model, restricted to the constraint surface, is then quantized. To eliminate the gauge arbitrariness of Eq. (A.8), gauge-fixing conditions that are second-class with respect to the γ's must be introduced. We assume a set of complete gauge-fixing conditions on $\mathcal{P}$, $\xi_a = 0$, such that $\{\xi_a, \gamma_b\}|_{C_a=0}$ has maximal rank.

Therefore, the canonical system with Hamiltonian H and a reduced phase space defined by $C_a = 0, \xi_a = 0$ removes all arbitrariness of motion and, in principle, generates a well-defined dynamics through a reduced Hamiltonian flow. Dirac showed that such a system obeys the reduced Hamiltonian

$$\bar{H} = H_{C_a,\xi_a=0}, \tag{A.9}$$

together with the reduced Poisson structure, known as the "Dirac bracket".

$$\{\cdot,\cdot\}_{\mathrm{DB}} = \{\cdot,\cdot\} - \{\cdot,\chi'_a\}X^{-1\,ab}\{\chi'_b,\cdot\}. \tag{A.10}$$

χ'_a's refer to all second-class constraints, including the gauge-fixing conditions ξ_a's, and X^{-1} is the inverse matrix of $\{\chi'_a, \chi'_b\}$. Motion on reduced phase space follows from

$$\frac{d}{dt}\cdot = \{\cdot, \bar{H}\}_{\mathrm{DB}}. \tag{A.11}$$

Reduced phase space quantization consists in quantizing Eqs. (A.9) and (A.10), that is, defining a Hilbert space $\mathcal{H}$, mapping Dirac brackets to commutators, and promoting $\bar{H}$ to a self-adjoint Hamiltonian operator on $\mathcal{H}$. Independent of the dBB construction, this approach is often plagued by mathematical difficulties and ambiguities from the starting point (see Chapter 13 in [48]), since quantization procedures are generally not unique. An additional source of complication arises from the fact that the Dirac bracket may be a non-trivial function of the conjugate variables, making the quantization of such expressions potentially ill-defined globally.

The implementation of the dBB framework further complicates the matter. As Bohm argued [47], the dBB model for a general potential term $V(q)$ in the Hamiltonian is compatible only with the guidance equation on configuration space if Born statistics are to be reproduced. This imposes additional restrictions on the quantization of $\bar{H}$ and the Dirac bracket, as it leaves us no choice but to work with the conjugate variables

q, p, where the q's are coordinates on $\mathcal{Q}$. Consequently, one cannot exploit unitarity by choosing an alternative representation of phase space that mixes q's and p's.

Despite these general difficulties, we shall show that such issues do not arise in the reduced phase space quantization of the specific model studied in Chapter 4—namely, electromagnetism. Starting from the constrained system Eqs. (4.3) and (4.7), we work in the Coulomb gauge:

$$A_0 \approx 0, \qquad \partial_a A_a \approx 0. \tag{A.12}$$

The second-class constraints satisfy the algebra

$$\begin{aligned} \{A_0(x), \Pi^0(y)\} &= \delta^3(x-y), \\ \{\partial_a A_a(x), \partial_b \Pi^b(y) - \rho(y)\} &= \Delta\, \delta^3(x-y). \end{aligned} \tag{A.13}$$

Thus, Dirac bracket Eq. (A.10) reads (removing the (A_0, Π^0) part)

$$\begin{aligned} \{\cdot,\cdot\}_{\mathrm{DB}} = \{\cdot,\cdot\} \quad &- \int_{x,y} \{\cdot, \partial_a A_a(x)\} \Delta^{-1}_{x,y} \{\partial_b \Pi^b(y), \cdot\} \\ &+ \int_{x,y} \{\cdot, \partial_a \Pi^a(x)\} \Delta^{-1}_{x,y} \{\partial_b A_b(y), \cdot\} \\ = \{\cdot,\cdot\} \quad &- \int_{x,y} \Delta^{-1}_{x,y} \partial_{x^a} \partial_{y^b} \{\cdot, A_a(x)\} \{\Pi^b(y), \cdot\} \\ &+ \int_{x,y} \Delta^{-1}_{x,y} \partial_{x^a} \partial_{y^b} \{\cdot, \Pi^a(x)\} \{A_b(y), \cdot\}, \end{aligned} \tag{A.14}$$

where $\Delta^{-1}_{x,y}$ is the inverse of the Laplace-Beltrami operator, defined as

$$\Delta \int_y \Delta^{-1}_{x,y} f_y = f_x, \tag{A.15}$$

which is a non-local operator in x and y. To complete the description of evolution on the reduced phase space, we recall that the constrained Hamiltonian is

$$\bar{H}[A^T_a, \Pi^{T\,a}, \bar{\psi}, \psi] = \int_x \left(\frac{1}{2} \Pi^a \Pi_a + \frac{1}{4} F^{ab} F_{ab} - A_a j^a + H_D \right)_{\partial_a A_a, \partial_a \Pi^a = 0}, \tag{A.16}$$

where A^T_a and $\Pi^{T\,a}$ are the transverse fields that coordinatize the reduced phase space.

Thus, we see that, fortunately, the Dirac bracket does not exhibit any problematic dependence on the conjugate variables. However, the reduced phase space formulation renders the canonical structure explicitly non-local due to the second term in the Dirac bracket. This non-locality is, of course, merely a by-product of gauge fixing—which is

a relatively common phenomenon—and does not affect the evolution of the physical transverse degrees of freedom.

We can quantize Eq. (A.16) by imposing the relations

$$\begin{aligned}[\hat{A}^T_a(x), \hat{\Pi}^{Tb}(y)] &= i\left(\delta^b_a \delta^3(x-y) + \partial_{x^a}\partial_{y^b}\Delta^{-1}_{x,y}\right), \\ [\hat{\psi}^\alpha(x), \hat{\psi}^\dagger_\beta(y)]_+ &= \delta^\alpha_\beta \delta^3(x-y),\end{aligned} \tag{A.17}$$

following Eq. (A.14), with all the other commutator and anti-commutators vanishing, same as before. Explicitly, this entails

$$\begin{aligned}\hat{\Pi}^{Ta}(x) &\equiv -i\int_y \left(\delta^a_b \delta^3(x-y) + \partial_{x^a}\partial_{y^b}\Delta^{-1}_{x,y}\right)\frac{\delta}{\delta A^T_b(y)} \\ &= -i\frac{\delta}{\delta A^T_a(x)} - i\int_y \partial_{x^a}\partial_{y^b}\Delta^{-1}_{x,y}\frac{\delta}{\delta A^T_b(y)}, \\ \hat{\psi}^\dagger_\alpha(x) &\equiv \frac{\delta}{\delta \psi_\alpha(x)}.\end{aligned} \tag{A.18}$$

It is possible to employ dBB by imposing realized transverse configurations $\bar{A}^T_a$ and the guidance equation as

$$\dot{\bar{A}}^T_a(x) = \frac{\operatorname{Im}\Psi^* \int_y \left(\delta^b_a \delta^3(x-y) + \partial_{x^a}\partial_{y^b}\Delta^{-1}_{x,y}\right)\frac{\delta}{\delta A^T_b(y)}\Psi}{\Psi^*\Psi}, \tag{A.19}$$

which is a non-local equation, as expected from the non-locality of Dirac bracket in the classical case. Note that

$$\partial_a \dot{\bar{A}}^T_a(x) = 0, \tag{A.20}$$

as expected from dynamical consistency with the gauge-fixing constraint.

A.4 Dirac approach

Dirac observed that quantization proceeds more smoothly for systems with first-class constraints than for those with second-class constraints. Consequently, a more effective approach is to work within an extended phase space in which all constraints arise from a set of first-class constraints, in contrast to the reduced phase space method, where all constraints are treated as second-class.

Assume a Hamiltonian H and a set of constraints $\gamma_a \approx 0$. The Dirac approach essentially consists of standard canonical quantization, supplemented by the imposition

of the time-independent operator equations

$$\hat{\gamma}_a \, |\psi\rangle = 0, \tag{A.21}$$

where $|\psi\rangle$ is defined on the Hilbert space $\mathcal{H}$. The wave functions that solve these equations can be evolved unitarily according to $\hat{U} = e^{-it\hat{H}}$, and they depend non-trivially only on the physical (gauge-invariant) degrees of freedom.

We now discuss the application to the dBB model. To ensure full consistency with the configuration space representation compatible with dBB, it is sufficient to assume that the first-class constraints γ_a generate gauge orbits that close on configuration space and that they are identical along the cotangent bundle, i.e., independent of the momenta. Algebraically, this requires that the constraints be linear in the momenta, of the form

$$\gamma_a = \alpha_a^b(q) p_b + \chi_a(q) \approx 0, \tag{A.22}$$

where α_a^b and χ_a's are functions on configuration space. The quantum equations Eq. (A.21) then take the simple form

$$-i\alpha_a^b(q) \frac{\partial}{\partial_b} \psi + \chi_a(q)\psi = 0, \tag{A.23}$$

which can be solved analytically.

Let us note that in the case of simple constraints $p_a \approx 0$, the corresponding equations imply that the wave function is independent of the associated configuration variable, i.e., $\partial_{q^a}\psi = 0$. In other words, the wave function is constant along gauge orbits, which in turn leads to a guidance field that is orthogonal to these orbits. Viewed as a modified Hamilton–Jacobi system, this corresponds to a system with undetermined constants of motion along the gauge directions.

The Dirac approach is applied to the dBB electromagnetic model constructed in Chapter 4.

Appendix B

Triad formulation

Let us rewrite ADM Hamiltonian in terms of triad variables. Consider one-form fields $e^i_a(x)$ (triads), indexed by $i = 1, 2, 3$ on the manifold, such that

$$g_{ab} = e^i_a e^j_b \delta_{ij}, \tag{B.1}$$

which fixes the triads up to local SO(3) action. The duals to e^i's are defined as

$$e^a_i e^j_a = \delta^j_i. \tag{B.2}$$

We denote the conjugate momenta to g_{ab} by p^{ab}. Let us define

$$\pi^a_i = \delta_{ij} e^j_b p^{ab}. \tag{B.3}$$

We will see that these are canonically conjugate to e^i's.

The geometric sector of ADM Hamiltonian is

$$\mathcal{H}_g[g,p] = \frac{1}{16\pi G}\int_x N\sqrt{|g|}\left[\frac{p^{ab}\left(g_{ac}g_{bd} - g_{ab}g_{cd}\right)p^{cd}}{2|g|} - (R[g] - 2\Lambda)\right]. \tag{B.4}$$

The kinetic term reads

$$\begin{aligned} p^{ab}\left(g_{ac}g_{bd} - g_{ab}g_{cd}\right)p^{cd} &= p^{ab}(e^i_a e^k_c e^j_b e^l_d \delta_{ik}\delta_{jl} - e^i_a e^k_b e^j_c e^l_d \delta_{ik}\delta_{jl})p^{cd} \\ &= \pi^a_i(e^j_a e^i_b - e^i_a e^j_b)\pi^b_j, \end{aligned} \tag{B.5}$$

where in the last line, Eq. (B.3) has been applied repeatedly. The determinant of the

metric is $|g| = |e|^2$ from definition. This leads to

$$\frac{1}{16\pi G}\int_x N\left[\frac{1}{2|e|}\pi_i^a\left(e_a^j e_b^i - \frac{1}{2}e_a^i e_b^j\right)\pi_j^b - (R[e]-2\Lambda)|e|\right], \tag{B.6}$$

which is Eq. (3.1).

Appendix C

Miscellaneous

C.1 Sim(3)

For particle mechanics, we consider the translation, rotation, and dilatation groups, as they are the building blocks of the similarity group. Here, we work with the action of the groups within a given coordinate system. We can then proceed to compute the generators of the Lie algebra explicitly. We assume that the configuration space of N particles in three-dimensional Euclidean space is given, coordinatized by the variables q_I^a, where $a = 1, 2, 3$ and $I = 1, 2, \ldots, N$.

Translations, T(3)

T(3) is isomorphic to $\mathbb{R}^3$. It is a three-dimensional Lie group. We can coordinatize it by assigning numbers $(\alpha^1, \alpha^2, \alpha^3)$ to each element. The product of two elements is represented by the sum of their coordinates. It then follows that the action of T(3) is

$$T\left((\alpha^1, \alpha^2, \alpha^3), q_I^a\right) = q_I^a + \alpha^a. \tag{C.1}$$

Thus, in this particular coordinate system, we can work out the generators on T(3) as

$$t_{ib}^a q_I^b = \delta_b^a, \tag{C.2}$$

where they are labeled by i.

Hence, the generators are

$$t_{ib}^a = \delta_b^a \partial_i. \tag{C.3}$$

Note that the generators depend only on the spatial indices, and not on the particle index. This is a physical requirement which ensures that all of the particles are subject to the same transformation.

SO(3)

The three-dimensional rotation group, characterized by orthogonal matrices with unit determinant, SO(3), rotates the spatial components of q_I^a. The group manifold of SO(3) is topologically compact and non-simply connected. Therefore, we cannot cover the whole group with a single coordinate system. One approach is to work with Euler parameters to represent different rotations. However, given our interest in the generators, we will simply assign three numbers (SO(3) is a three-parameter group) to its elements. Thus, for rotations we have

$$T\left((\omega^1,\omega^2,\omega^3),q_I^a\right)=\Omega(\omega)_b^a q_I^b, \tag{C.4}$$

where Ω is the corresponding rotation matrix. Hence,

$$t_{ib}^a q_I^b = \frac{d}{d\omega^i}\Omega_b^a\Big|_{\omega=0} q_I^b. \tag{C.5}$$

By a straightforward calculation, it can be shown that the derivative of Ω around the identity is an antisymmetric matrix with entries 1 and 0. Using the Levi-Civita symbol, we have

$$t_{ib}^a q_I^b = \delta_{iq}\epsilon^{aqs}\delta_{sr}q_I^r. \tag{C.6}$$

Eliminating q_I^b, we arrive at

$$t_{ib}^a = \delta_{iq}\epsilon^{aqp}\delta_{br}q_I^r\partial_p. \tag{C.7}$$

$\mathbb{R}^+$

The dilatation group rescales the system and is isomorphic to the set of positive real numbers. It is a one-dimensional group, and its elements can be represented by a single number α. Its action is given by

$$T\left(\alpha,q_I^a\right)=\alpha q_I^a. \tag{C.8}$$

Thus, the generators are

$$t^a_{ib} q^b_I = \delta^a_b q^b_I, \tag{C.9}$$

leading to

$$t^a_{ib} = \delta^a_b q^m_I \partial_m. \tag{C.10}$$

The three groups studied here comprise the similarity group Sim(3). The generators of each one of these groups is summarized in table C.1.

group	**index**	**generator** $(t^a_{\alpha b})$
translations, T(3)	$\alpha = i = 1,2,3$	$\delta^a_b \partial_i$
rotations, SO(3)	$\alpha = j+3 = 4,5,6$	$\epsilon^{ajp}\delta_{br} q^r \partial_p$
dilatations, $\mathbb{R}^+$	$\alpha = 7$	$\delta^a_b q^m \partial_m$

TABLE C.1: The generators of the groups SO(3), T(3), $\mathbb{R}^+$

C.2 CONSTRUCTION OF THE SHAPE METRIC IN $\mathbb{R}^{3N}$

Let us use the explicit form of the generators to construct the shape metric introduced in Eq. (2.23). As $\mathcal{S} = \mathbb{R}^{3N}/\mathrm{Sim}(3)$, we can represent each element of the shape space q_s by a group orbit in $\mathbb{R}^{3N}$ as

$$q_s \to T(\alpha, q^a_I), \tag{C.11}$$

where $T(\alpha, \cdot)$ denotes an arbitrary action of Sim(3). Therefore, for variations on $\mathcal{S}$, we have the arbitrary action of the Lie algebra:

$$\delta q_s \to \delta q^a_I + \delta\alpha^c t^a_{cb} q^b_I. \tag{C.12}$$

The definition of the shape metric in Eq. (2.23) yields

$$\begin{aligned} ds^2 = g_{ab}\delta q^a_s \delta q^b_s &= \min_{\delta\alpha} g^{IJ}_{ab} \left(\delta q^a_I + \delta\alpha^c t^a_{cd} q^d_I\right)\left(\delta q^b_J + \delta\alpha^c t^b_{cd} q^d_J\right) \\ &= \min_{\delta\alpha} g^{IJ}_{ab} \left(\delta q^a_I \delta q^b_J + \delta\alpha^c \delta q^b_J t^a_{cd} q^d_I + \delta\alpha^c \delta q^a_I \delta t^b_{cd} q^d_J\right), \end{aligned} \tag{C.13}$$

where it can be solved for α, given any configuration assigned to the orbit represented by q_s. It can be shown that the minimum occurs when the variations are orthogonal to the gauge orbits [34].

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